Mathematical Understanding: A Philosophical Perspective

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The problem of multiple proofs

On the standard account, the value of a mathematical proof is that it warrants the truth of the resulting theorem.

Why, then, do we often value a new proof of a previous established theorem?

For example, Gauss published six proofs of the law of quadratic reciprocity in his lifetime, and left us two unpublished versions as well.

Franz Lemmermeyer's used to have a list of 246 published proofs online.

The problem of multiple proofs

I began the 2006 paper "Mathematical Method and Proof" with this question.

That same year, John Dawson published "Why do mathematicians re-prove theorems?"

See also his *Why Prove It Again?* (2015).

The problem of multiple proofs

The central question: what do we get from a proof beyond the knowledge that the resulting theorem is true?

Dialectic:

- Consider three mathematical theorems.
- Consider multiple proofs of each.
- Explore intuitions as to what we get from them.
- Use that to motivate, and lay groundwork for, a theory of mathematical understanding.

Sums of squares

In the *Arithmetic*, Diophantus notes that

- $5 = 2^2 + 1^2$
- $13 = 3^2 + 2^2$
- $5 \times 13 = 65 = 8^2 + 1^2 = 7^2 + 4^2$.

Theorem. If x and y can each be written as a sum of two integer squares, then so can xy.

Sums of squares

Proof #1. Suppose $x = a^2 + b^2$, and $y = c^2 + d^2$. Then

$$xy = (ac - bd)^2 + (ad + bc)^2,$$

a sum of two squares. □

In more detail:

$$\begin{aligned}
&(ac - bd)^2 + (ad + bc)^2 \\
&\qquad = a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2d^2 \\
&\qquad = a^2c^2 + b^2d^2 + a^2d^2 + b^2d^2 \\
&\qquad = (a^2 + b^2)(c^2 + d^2)
\end{aligned}$$

Note: $(ac + bd)^2 + (ad - bc)^2$ works just as well.

Sums of squares

Define the *Gaussian integers*:

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

If $\alpha = u + vi$, define the *conjugate*:

$$\overline{\alpha} = u - vi.$$

We have $\overline{\alpha\beta} = \overline{\alpha} \cdot \overline{\beta}$.

Define the norm:

$$N(\alpha) = \alpha\overline{\alpha} = (u + iv)(u - iv) = u^2 - i^2v^2 = u^2 + v^2.$$

Then

$$N(\alpha\beta) = \alpha\beta \cdot \overline{\alpha\beta} = \alpha \cdot \beta \cdot \overline{\alpha} \cdot \overline{\beta} = \alpha\overline{\alpha} \cdot \beta\overline{\beta} = N(\alpha)N(\beta).$$

Sums of squares

Proof #2. Suppose $x = N(\alpha)$ and $y = N(\beta)$ are sums of two squares. Then $xy = N(\alpha\beta)$, a sum of two squares. $\square$

Things to like about this example:

- It's accessible.
- It's substantial.
- We have good intuitions.
- It's clear that something interesting is going on.

Other motivating questions

- the nature of historical advances
- the nature of diagrammatic reasoning
- the role of computers in mathematical proof
- the reliability of mathematical reasoning
- the role of abstraction in mathematical reasoning

Intuitions

Mathematics is hard.

Mathematical solutions, proofs, and calculations involve long sequences of steps, that have to be chosen and composed in precise ways.

To compound matters, there are too many options; among the many steps we may plausibly take, most will get us absolutely nowhere.

And we have limited cognitive capacities — we can only keep track of so much data, anticipate the result of a few small steps, remember so many background facts.

We rely on our understanding to help us and to guide us.

Methods and abilities

Mathematical knowledge is often cast as *propositional* knowledge, like definitions and theorems.

But understanding seems to require a kind of *procedural* knowledge.

Knowledge is static, understanding is dynamic.

Understanding guides thought.

It's the capacity to think and reason mathematically.

Methods and abilities

One approach: talk about *methods*, i.e. heuristic, fallible, procedures for solving problems, searching for proofs, verifying inferences, etc.

Straightforward model:

- We face tasks (solving a problem, proving a theorem, verifying an inference, developing a theory, forming a conjecture).
- "Reasoning" involves passage though various epistemic states.
- "Understanding" (methods, techniques, procedures, protocols, tactics, strategies, . . .) makes this passage possible.

Methods and abilities

Talk of methods may be too fine-grained.

People multiply numbers in different ways. Sometimes we only care about the ability to do so.

Another approach: talk about *abilities*, or capacities for thought.

Methods and abilities

Understanding involves:

- Being able to recognize the nature of the objects and questions before us.
- Being able to marshall the relevant background knowledge and information.
- Being able to traverse space the of possibilities before us in a fruitful way.
- Being able to identify features of the context that help us cut down complexity.

Abilities *give rise* to observable behavior.

They constitute functional models that *explain* it.

Methods and abilities

In her introduction, Tania distinguished between:

- ability-based accounts: behavioral descriptions
- model-based accounts: explanations in terms of mental representations

I am not advocating for the first! However, representations are useless without the ability to use them.

I am distinguishing between:

- Static accounts: having knowledge, representations, networks
- Dynamic accounts: having the ability to use these to reason, solve problems, prove theorems, etc.

Methods and abilities

The methodological thesis: for many purposes, we do not need anything more than an account of the abilities, or capacities, that we take to be constitutive of particular instances understanding.

To characterize a particular type of understanding, it suffices to characterize the abilities it confers.

If this explains the data (mathematical practice) and is scientifically useful, we need not look any further.

See also Janet Folina, "Towards a better understanding of mathematical understanding."

Methods and abilities

We can apply this point of view wherever talk of understanding arises:

- contemporary mathematics
- mathematical education
- history of mathematics
- automated reasoning and AI

To do that, we need better ways of talking about methods and abilities.

Mathematical understanding

Some things I have written:

- Mathematical method and proof (2006)
- Understanding proof (2008)
- Understanding, formal verification, and the philosophy of mathematics (2010)
- Modularity in mathematics (2020)
- Reliability of mathematical inference (2021)
- The design of mathematical language (2024)

Most of them contain formal code snippets.

All of them contain historical examples.

Mathematical understanding

I have also done work on:

- Applications of logic to mathematics.
- Euclidean diagrammatic reasoning.
- History of nineteenth century number theory.
- Interactive proof assistants.
- Automated reasoning.
- Machine learning for mathematics.

These have all been informed by thinking about mathematical understanding.

Concepts

In the psychological literature, concepts are sometimes thought of in terms of categorization (e.g. prototypes and exemplars).

From a logical perspective, a concept is given by a definition, in a suitable formal language.

These don't work so well for the philosophy of mathematics.

What does it mean to understand the concept of a *group*? Or the concept of a *function*? Or the concept of a *Riemannian manifold*?

Concepts

Mathematical concepts have some interesting features:

- Membership is often sharply defined.
- Mathematical concepts evolve over time.
- Understanding a concept admits degrees.
- Various things can “improve our understanding” of a concept.
- One can speak of implicit uses of a concept.

Concepts

One solution: think of a mathematical concept as a bundle of abilities.

For example, understanding the group concept includes:

- Knowing the definition of a group.
- Knowing common examples of groups, and being able to recognize implicit group structures when it is fruitful to do so.
- Knowing how to construct groups from other groups or other structures, in fruitful ways.
- Recognizing that there are different kinds of groups (abelian, nilponent, solvable, finite vs. infinite, continuous vs. discrete) and being able/prone to make these distinctions.
- Knowing various theorems about groups, and when and how to apply them.

Concepts

This renders "the group concept," for example, vague and open-ended.

But the notion *is* vague and open-ended:

- We can talk about student understanding.
- We can talk about the role of the concept in contemporary mathematics.
- We can talk about the historical development.

The proposal suggests that we can make our talk more precise by being more precise about the abilities (or methods, or capacities) we have in mind.

Goals

We are looking for a philosophical account that:

- is clear, precise, and internally coherent
- accords with our intuitions
- fits the data (what we see in mathematics)
- can inform (and can be informed by) other pursuits:
  - history of mathematics
  - interactive theorem proving and automated reasoning
  - psychology and cognitive science
  - mathematics education
  - mathematics itself

Toward a theory of mathematical understanding

General strategy: think globally, act locally.

Keep the big questions in mind, but address more focused questions:

- What kinds of things can be inferred from the diagrams in Euclid's *Elements*?
- How did Dedekind's introduction of *ideals*, or Dirichlet's introduction of *characters*, contribute to number theory?
- What mechanisms can be used to model algebraic hierarchies in interactive proof assistants?

Toward a theory of mathematical understanding

If we

- continue to make progress on specific questions and
- keep the general questions in mind,

a theory of mathematical understanding will eventually emerge.

What about the overarching questions:

- Why do we care about understanding?
- Why do we do mathematics the way we do?

From formal methods to epistemology

View mathematics as a communal practice designed to meet fundamental constraints:

- scientific utility
- cognitive efficiency
- communicability
- reliability
- stability

The best justification for mathematics is that it serves its purposes well.

We need to better understand how and why.