Methodology and metaphysics in the development of Dedekind's theory of ideals

Jeremy Avigad

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1 Introduction

Philosophical concerns rarely force their way into the average mathematician's workday. But, in extreme circumstances, fundamental questions can arise as to the legitimacy of a certain manner of proceeding, say, as to whether a particular object should be granted ontological status, or whether a certain conclusion is epistemologically warranted. There are then two distinct views as to the role that philosophy should play in such a situation.

On the first view, the mathematician is called upon to turn to the counsel of philosophers, in much the same way as a nation considering an action of dubious international legality is called upon to turn to the United Nations for guidance. After due consideration of appropriate regulations and guidelines (and, possibly, debate between representatives of different philosophical factions), the philosophers render a decision, by which the dutiful mathematician abides.

Quine was famously critical of such dreams of a "first philosophy." At the opposite extreme, our hypothetical mathematician answers only to the subject's internal concerns, blithely or brashly indifferent to philosophical approval. What is at stake to our mathematician friend is whether the questionable practice provides a proper mathematical solution to the problem at hand, or an appropriate mathematical understanding; or, in pragmatic terms, whether it will make it past a journal referee. In short, mathematics is as mathematics does, and the philosopher's task is simply to make sense of the subject as it evolves and certify practices that are already in place. In his textbook on the philosophy of mathematics (Shapiro, 2000), Stewart Shapiro characterizes this attitude as "philosophy last, if at all."

The issue boils down to whether fundamental questions as to proper mathematical practice should be adjudicated with respect to general, and potentially extra-mathematical, considerations, or with respect to inherently "mathematical" standards, values, and goals. Of course, what typically happens lies somewhere in between. Mathematics is not a matter of "anything goes," and every mathematician is guided by explicit or unspoken assumptions as to what counts as legitimate — whether we choose to view these assumptions as the product of birth, experience, indoctrination, tradition, or philosophy. At the same time,

mathematicians are primarily problem solvers and theory builders, and answer first and foremost to the internal exigencies of their subject.

It seems likely, then, that any compelling philosophical account of mathematics will have to address both general philosophical principles and the more pragmatic goals of the practice. For example, many today hold that ontological questions in the sciences are to be adjudicated holistically, with respect to (possibly competing) standards such as generality, simplicity, economy, fecundity, and naturalness.[1] When it comes to mathematics, it is hard to see how one can take this dictum seriously without first gaining some clarity as to the ways that the objects of modern mathematical discourse influence the subject along these axes.

The work of Richard Dedekind provides a clear example of interplay between general philosophical views and methodological concerns. His work has certainly had a tangible effect on mathematics, inaugurating practices that were to proliferate in the decades that followed. These include the use of infinitary, set-theoretic language; the use of non-constructive reasoning; axiomatic and algebraic characterization of structures, and a focus on properties that can be expressed in terms of mappings between them; the use of particular algebraic structures, like modules, fields, ideals, and lattices; and uses of algebraic mainstays like equivalence relations and quotient structures. These innovations are so far-reaching that it is hard not to attribute them to a fundamentally different conception of what it means to do mathematics. Indeed, we find Dedekind addressing foundational issues in his *Habilitationsrede* (1854), and his constructions of the real numbers (1872) and the natural numbers (1888) are well-known. But even his distinctly mathematical writings bear a strongly reflective tone, one that is further evident in his correspondence with colleagues; see (Dedekind, 1968; Dugac, 1976; Ewald, 1996; van Heijenoort, 1967).

In recent years, Dedekind has received a good deal attention from both philosophers and historians of mathematics, e.g. (Corry, 2004; Dugac 1976; Ferreirós, 1999; Haubrich, 1992; Nicholson, 1993; Reck, 2003; Sieg and Schlimm, to appear; Stein 1988). It is interesting to compare the various terms that are used to characterize his work. For example, Gray attributes the success of Dedekind's theory of ideals to its handling of ontological issues:

> It occurred in a leading branch of contemporary mathematics, the algebraic theory of numbers. This gave it great weight as an example of how existence questions could be treated. It arose from a genuine question in research mathematics, the outcome of which necessarily involved a point of mathematical ontology. (Gray, 1992, 233)

Here, Gray is using the word "ontology" in a distinctly methodological sense. In contrast, Stein proclaims that Dedekind is "quite free of the preoccupation with 'ontology' that so dominated Frege, and has so fascinated later philosophers" (Stein, 1988, 227), and praises Dedekind instead for the way he was able to "open up the possibilities for developing concepts." Edwards finds Dedekind's work

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[1](Maddy, 2005) nicely surveys the use of terms like these by Quine.

guided by "strong philosophical principles," judging salient characteristics of his work to be "dictated by [his] set-theoretic prejudices" (1980, 321). According to Corry,

> Dedekind's overall mathematical output reflects a remarkable methodological unity, characterized, above all, by a drive to radically reformulate the conceptual settings of the mathematical problems he addressed, through the introduction and improvement of new, more effective, and simpler concepts. (Corry, 2004, 68)

Corry favors the notion of an "image" of mathematics to describe the associated world view.

I believe it is fruitless to debate whether Dedekind's metaphysics or his methodology is philosophically prior; that is, whether we should take his general conception of mathematics to explain his methodological innovations, or vice-versa. In the end, the two cannot be so cleanly separated. But philosophical discussions of Dedekind tend to focus on the metaphysical side, for example, on the ontological ramifications of his uses of set-theoretic methods, or on the forms of structuralism or logicism that are implicit in his views; see, for example, (Ferreirós, 1999, Chapter VII) and (Reck, 2003). My goal here is to balance these with a discussion of some of the methodological aspects of Dedekind's work. In other words, I will try to clarify some of the more distinctly mathematical concerns Dedekind faced, and explore the mathematical ramifications of the methods he introduced.[2]

I will focus, in particular, on his development of the theory of ideals. Towards the end of the 1850's, both Dedekind and Leopold Kronecker aimed to extend Ernst Kummer's theory of ideal divisors from cyclotomic cases to arbitrary algebraic number fields. Dedekind published such a theory in 1871, but he continued to modify and revise it over the next 23 years, publishing three additional versions during that time. Kronecker's theory was published in 1882, although it seems to have been developed, for the most part, as early as 1859. Despite the common starting point, the two theories stand in stark contrast to one another, representing very different sets of mathematical values. These parallel developments are therefore a wonderful gift of history to philosophers and mathematicians alike. Paying attention to the substance of Dedekind's revisions, and to the differences between his theory and Kronecker's, can illuminate important mathematical issues and help us understand some of the goals that drive the subject.

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[2]To be sure, historical treatments of Dedekind often discuss such methodological aspects of his work; see, for example, (Ferreirós, 1999, Chapter III) or (Haubrich, 1992, Chapter 1). Here, I simply explore some of these issues in greater depth.

Jeremy Gray has recently brought to my attention a book by David Reed (1995), which includes a lengthy discussion of the development of algebraic number theory in the hands of Dedekind and Kronecker. Although there is a good deal of overlap, Reed's presentation is largely complementary to mine. In particular, whereas I focus on the relationship to prior mathematical developments, Reed emphasizes differences between Dedekind and Kronecker vis-à-vis the subsequent development of class field theory.

Below, then, is a preliminary survey of some of the epistemological values that are evident in the development of Dedekind's theory. The analysis is rough, and should be extended in both depth and breadth. Towards greater depth, we should strive for a more careful and precise philosophical characterization of these values, and submit them to closer scrutiny. In particular, it is important to recognize that Dedekind's methodological innovations are not uniformly viewed as positive ones. In Section 3 below, I will discuss some of the aspects of his work that were controversial at the time, and serve to distinguish it from Kronecker's. It will become clear that I am sympathetic to the claim that something important has been lost in the transition to modern mathematics, and so my enumeration of some of the purported benefits of Dedekind's revisions should not be read as wholesale endorsement. A broader philosophical analysis would benefit from contrasting the work of Dedekind and Kronecker, and tracing the influence of and interactions between their differing mathematical styles, through the twentieth century, to the present day.[3]

2 The need for a theory of ideal divisors

In this section, I will sketch the historical circumstances that gave rise to Dedekind's theory of ideals.[4] Many of the original works I will mention below are accessible and well worth reading. But there are a also number of good secondary references to the history of nineteenth century number theory, including (Edwards 1996) and (Goldman, 1998) and Stillwell's notes to (Dedekind, 1877) and (Dirichlet, 1863). Edwards' excellent article (1980) focuses specifically on the development of the theory of ideals, and the topic is also covered in some detail in (Corry, 2004). My goal here is not to extend this historical scholarship, but, rather, to supplement it with a more careful discussion of the methodological issues.[5]

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[3]Edwards' detailed studies of nineteenth century number theory, and of the development of ideal theory in particular, provide a strong starting point; see (Edwards 1980, 1983, 1989, 1990, 1992, 1996).

[4]Kummer called the ideal divisors described below "ideal complex numbers," Kronecker called them "divisors," and Dedekind associated them with the set-theoretic notion of an "ideal." I will use the term "ideal divisor" generically, to include all three notions.

[5]Since my goals are not primarily historical, I will generally use contemporary terms to describe the mathematical substance of the developments. For example, where I speak of the ring of integers in a finite extension of the rationals, Dedekind refers to the "system" of

The problem of determining whether various systems of equations have solutions in the integers, and of finding some or all of the solutions, dates back to antiquity. In his *Algebra*, Leonhard Euler (1770) took the bold step of using complex numbers towards those ends. For example, in articles 191–193 of Part II, Euler considers the question as to when an expression of the form $x^2 + cy^2$ can be a perfect cube, where c is fixed and x and y range over integers. Euler first notes that the expression can be factored as $(x + y\sqrt{-c})(x - y\sqrt{-c})$. Assuming $-c$ is not a perfect square, Euler asserts that when the two multiplicands have no common factor, in order for the product to be a cube, each factor must itself be the cube of an number of the form $p + q\sqrt{-c}$, where p and q are integers. If $x + y\sqrt{-c}$ is equal to $(p + q\sqrt{-c})^3$ for some p and q, then $x - y\sqrt{-c}$ is equal to $(p - q\sqrt{-c})^3$. Expanding the products and setting real and imaginary parts equal enables Euler to show, for example, that $x^2 + 2$ is a cube only when $x = \pm 5$, and $x^2 + 4$ is a cube only when $x = \pm 2$ or $x = \pm 11$.

There is a gap in Euler's argument.[6] It lies in the assertion that "integers" of the form $x + y\sqrt{-c}$ behave like ordinary integers, in the sense that if the product of two numbers of this form is a cube, and the two have no nontrivial common factor, then each must itself be a cube. For the ordinary integers, this follows from the unique factorization theorem, since, for the product of two such numbers to be a cube, any prime divisor of either factor must occur in the product a multiple of three times, and hence in that factor as well. In more general rings, a nonzero element c is said to be *irreducible* if it cannot be written as a product ab where neither a nor b is a unit (i.e. a divisor of 1). An integer c that is neither zero nor a unit is said to be *prime* if whenever c divides a product, ab, it divides either a or b. Unique factorization for the ordinary integers follows from the fact that in this setting, these two notions coincide.

In more general rings of integers, however, this need not be the case. For example, among complex numbers of the form $x + y\sqrt{-5}$, 6 can be written as either $(1 + \sqrt{-5})(1 - \sqrt{-5})$ or $2 \cdot 3$, and each of these factors is irreducible. The fact that 3 divides the product $(1 + \sqrt{-5})(1 - \sqrt{-5})$ but does not divide either factor shows that 3 is not prime. Furthermore, the perfect square 9 can be written as the product $(2 + \sqrt{-5})(2 - \sqrt{-5})$, even though neither $2 + \sqrt{-5}$ nor $2 - \sqrt{-5}$ is a square of a number of the form $x + y\sqrt{-5}$, nor do they share a nontrivial factor of that form.

In (1828), Gauss published a proof of unique factorization for what we now call the "Gaussian integers," that is, the ring of numbers of the form $\{x + yi \mid x, y \in \mathbb{Z}\}$. This proof appeared in the context of his study of biquadratic residues, i.e. residues of integer primes raised to the fourth power. Thus Euler's and Gauss's work made two things clear:

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integers in such a field. Readers interested in terminological nuances and historical context should consult the sources cited above. That said, one thing that is striking about Dedekind's presentations is their proximity to modern ones, even down to terminology and notation. For example, we still use his terms "module" and "ideal"; and, like Dedekind, we generally use Greek letters $\alpha, \beta, \gamma, \ldots$ to range over algebraic integers, and Fraktur letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ to range over ideals.

[6]The error is puzzling, especially since in other places Euler seems to know better. See the discussion in (Edwards, 1996, Section 2.3).

- Various rings of numbers extending the integers are useful in the study of fundamental questions involving equations and congruences in the ordinary integers.

- Unique factorization is an important feature of some such rings, and has a bearing on the original questions.

Aiming to generalize Gauss's results and obtain higher reciprocity laws, Ernst Kummer studied rings of numbers of the form $a_0 + a_1\zeta + a_2\zeta^2 + \ldots + a_{p-1}\zeta^{p-1}$, where ζ is a primitive pth root of unity.[7] We now call these *cyclotomic integers*. By 1844, he knew that unique factorization can fail in such rings; the first case occurs when $p = 23$ (for details, see (Edwards, 1980 or 1996)). In 1846, Kummer published a theory which aimed to remedy the situation through "the introduction of a peculiar sort of imaginary divisors" which he called "ideal complex numbers." The paper begins with the observation that in rings of cyclotomic integers with prime exponent, even when a number cannot be decomposed into factors,

> ... nonetheless, it may not have the true nature of a complex prime number, since it lacks the first and most important property of the prime numbers: namely, that the product of two prime numbers is not divisible by any prime different from them. Although they are not decomposible into complex factors, such numbers have, nonetheless, the nature of composite numbers; the factors are thus not actual, but *ideal complex numbers*. (Kummer, 1846, 319; my translation)

Given that unique factorization can fail in such a ring, the goal is to find a way to reason about the factorization of an element into ideal primes.

> In order to arrive at a firm definition of the true (generally ideal) prime factors of complex numbers, it was necessary to identify those properties of the complex numbers which persist in all circumstances, and are independent of whether it happens that the actual decomposition exists. ... Many of these persistant properties of the complex numbers are suitable for use in defining the ideal prime factors and, basically, always yield the same result. Of these, I have chosen the simplest and most general. (*ibid.*, 320)

The idea is this. Given a ring of cyclotomic integers that *does* satisfy unique factorization, and given a prime element α of that ring, one can characterize the property of divisibility of an element x by α in terms which do not mention α directly. In fact, Kummer showed that one can find such "divisibility tests" $P_\alpha(x)$ that make sense even in rings that *do not* satisfy unique factorization, and

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[7] A number can be represented by more than one expression of this form. Since $\zeta^p - 1 = (\zeta - 1)(\zeta^{p-1} + \zeta^{p-2} + \ldots + 1) = 0$, we have $\zeta^{p-1} + \zeta^{p-2} + \ldots + 1 = 0$; and then ζ^{p-1}, and hence the sum $a_0 + a_1\zeta + a_2\zeta^2 + \ldots + a_{p-1}\zeta^{p-1}$, can be expressed in terms of smaller powers of ζ.

possess all the central features of the usual notion of "divisibility by a prime." Thus one can think of these $P_\alpha(x)$ as representing the property of divisibility by an ideal prime factor α, even in cases where $P_\alpha(x)$ does not coincide with divisibility by any *actual* prime in the ring. Kummer was, in particular, able to show that unique factorization holds, in general, when expressed in terms of these predicates.[8] Thus

> ...it follows that calculation with complex numbers through the introduction of the ideal prime factors becomes exactly the same as calculations with the integers and their actual integer prime factors. (*ibid.*, 323)

So Kummer's "definition" of the ideal complex numbers amounts to an explicit, algorithmic description of the associated divisibility predicates. Even though Kummer described these ideal numbers as "peculiar," he seems perfectly comfortable with this manner of definition, comparing it to the introduction of complex numbers in algebra and analysis, and to the introduction of ideal elements in geometry.[9] He is forceful in emphasizing the importance of these ideal elements to the study of the cyclotomic integers, and hence to number theory more generally:

> ...one sees that the ideal factors unlock the inner nature of the complex numbers, make them, as it were, transparent, and show their inner crystalline structure. (*ibid.*, 323)

Richard Dedekind and Leopold Kronecker later took up the task of extending the theory to the rings of integers in *arbitrary* finite extensions of the rationals. (See the beginning of Section 4 for a clarification of the notion of "integer" in this context.) Despite their common influences and goals, however, the theories they developed are quite different, in ways which I will discuss in Section 3.

Dedekind ultimately published four versions of his theory of ideals (1871, 1877, 1879, 1894). The versions of 1871, 1879, and 1894 appeared, respectively, in his "supplements," or appendices, to the second, third, and fourth editions of his transcription of Dirichlet's lectures on number theory (Dirichlet, 1863). The remaining version was written at the request of Lipschitz, translated into French, and published in the *Bulletin des Sciences Mathématiques et Astronomiques* in 1876–1877. It was also published as an independent monograph in 1877, and is, in essence, an expanded presentation of the version he published in 1879.

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[8] More precisely, Kummer also had to define the notion of divisibility by *powers* of the ideal prime divisors. The *product* of powers of distinct primes could then be understood in terms of their least common multiple.

[9] Kummer refers specifically to the introduction of the "ideal common chord" of two non-intersecting circles. Such uses of the term "ideal" in geometry are due to (Poncelet, 1822); see (Rosenfeld, to appear).

3 Dedekind's emphasis on conceptual reasoning

Below we will discuss a number of aspects of Dedekind's theory of ideals. There is one, however, that gives it a character that is squarely opposed to that of Kronecker's theory, namely, Dedekind's emphasis on "conceptual" over algorithmic reasoning.[10]

Whereas Kronecker's theory of ideal divisors is explicitly algorithmic throughout,[11] Dedekind's stated goal was to *avoid* algorithmic reasoning:

> Even if there were such a theory, based on calculation, it still would not be of the highest degree of perfection, in my opinion. It is preferable, as in the modern theory of functions, to seek proofs based immediately on fundamental characteristics, rather than on calculation, and indeed to construct the theory in such a way that it is able to predict the results of calculation...(Dedekind, 1877, *Werke* vol. 3, 296; trans. Stillwell, 1996, 102; also translated in Stein, 1988, 245)

In this passage, Dedekind is referring to Riemann's approach to the theory of functions of a complex variable, in which functions are characterized by their topological and geometric properties.[12] Dedekind makes a similar claim in a letter to Lipschitz, written around the same time:

> My efforts in number theory have been directed towards basing the work not on arbitrary representations or expressions but on simple foundational concepts and thereby — although the comparison may sound a bit grandiose — to achieve in number theory something analogous to what Riemann achieved in function theory, in which connection I cannot suppress the passing remark the Riemann's principles are not being adhered to in a significant way by most writers — for example, even in the newest work on elliptic functions. Almost always they mar the purity of the theory by unnecessarily bringing in forms of representation which should be results, not tools, of the theory. (Dedekind, 1876, *Werke* vol. 3, 468–469; quoted and translated in Edwards, 1983)

Dedekind returns to this point in (1895), an essay we will discuss below. There, he first quotes an excerpt from Article 76 of Gauss's *Disquisitiones Arithmeticae* (1801), in which Gauss observes that Wilson's theorem was first published by Waring. In that excerpt, Gauss notes that

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[10] (Ferreirós, 1999) explores some of the historical developments that may have contributed to this aspect of Dedekind's thought; see, in particular, Chapter I, Section 4.

[11] The centrality of algorithms is clear not only in Kronecker's mathematical work, but in the only foundational essay he published, (Kronecker, 1887). See also (Edwards, 1989) for a discussion of Kronecker's foundational views.

[12] See (Ferreirós, 1999) and (Laugwitz, 1999) for discussions of Riemann's work that emphasize features that Dedekind may have had in mind.

> ... neither of them was able to prove the theorem, and Waring con-
> fessed that the demonstration was made more difficult because no
> *notation* can be devised to express a prime number. But in our
> opinion truths of this kind should be drawn from the ideas involved
> rather than from notations. (Gauss, 1801, article 76; trans. Clarke,
> 1966. Dedekind, 1895, quotes the original Latin.)

Dedekind goes on:

> When one takes them in the most general sense, a great scientific
> thought is expressed in these words, a decision in favor of the inter-
> nal [*Innerliche*], in contrast to the external [*Äußerlichen*]. This con-
> strast is repeated in almost every area of mathematics; one need only
> think of the theory of [Complex] functions, and *Riemann*'s definition
> of functions through internal characteristic properties, from which
> the external forms of representation necessarily arise. (Dedekind,
> 1895, *Werke* vol. 2, 54–55; my translation)

When it comes to making sense of the passages above, it is easier to say what
Dedekind is trying to avoid: definitions of mathematical objects and systems of
objects in terms of symbolic expressions and methods of acting upon them (for
example, in thinking of functions as given by *expressions* of a certain sort), and
proofs that rely on a particular choice of representation when many equivalent
representations are available. Saying, in a positive way, how the new methods of
reasoning are supposed to accomplish this takes more effort. Dedekind's math-
ematical work, however, provides us with a sense of what he had in mind. For
one thing, his foundational essays (Dedekind 1854, 1872, 1888) all focus on ax-
iomatic characterization of structures of interest; and in (Dedekind, 1888) there
is a proof that his axiomatization of the natural numbers is categorical, i.e. it
characterizes the structure uniquely, up to isomorphism. We will see below that
this focus on axiomatic characterization is central to his work. We will also see
Dedekind embrace the ability to introduce new mathematical structures and op-
erations on these structures using set-theoretic definitions, without concern for
the way that the elements of these structures are to be represented syntactically,
and therefore without providing algorithms that mirror the set-theoretic opera-
tions. Indeed, Dedekind shows no qualms in taking the elements of a structure
to be infinitary objects in their own right, or referring, in a definition, to the
totality of subsets of an infinite domain. It is clear that Dedekind sees these
methods as part of the "conceptual" approach as well.

It is worth emphasizing that it is not axiomatic methods in and of themselves
that distinguish Dedekind's work from Kronecker's. In extending Kummer's
work on ideal divisors of the cyclotomic integers, Kronecker (1870) himself gave
an early an influential axiomatization of the notion of a group. Referring to
Gauss's classification of binary quadratic forms, he wrote:

> The extremely simple principles on which Gauss's method rests can
> not only be applied in the place mentioned above, but also in many

others and, in particular, already in the most elementary parts of
number theory. This fact suggests, and it is easy to convince oneself
of it, that these principles belong to a more general and more ab-
stract realm of ideas. It seems therefore to be appropriate to free the
further development of the latter from all inessential restrictions, so
that one is then spared from having to repeat the same argument in
the different cases of application. The advantage comes to the fore
already in the development itself, and the presentation (if it is given
in the most general way possible) thereby gains in simplicity and
clarity, since it clearly exhibits what alone is essential. (Kronecker,
1870, *Werke* vol. 1, 274–275; trans. Schlimm, 2005; also translated
in Wussing, 1984, 64)

Below we will see Dedekind offer similar pronouncements as to the simplicity
and generality to be gained from algebraic methods. So the differences between
the two do not lie in the use of algebraic notions *per se*, but, rather, in the
manner in which these notions are employed.

The progression from Dedekind's first theory of ideals to his last represents a
steady transition from Kummer's algorithmic style of reasoning to a style that
is markedly more abstract and set theoretic. Thus, it is not surprising that
Edwards, who laments mathematics' departure from the explicitly algorithmic
styles of Gauss, Kummer, and Kronecker, judges Dedekind's first version of ideal
theory to be his best (Edwards 1980, 1992). In contrast, Emmy Noether, who
inherited the mantle of structuralism from Dedekind through Hilbert, expressed
a clear preference for the last (see McLarty's contribution to this volume). Trac-
ing the development of Dedekind's thinking can therefore help us understand
what is at stake.

As noted in the introduction, this essay focuses on Dedekind's point of view.
This point of view was controversial at the time, and since then there has
been a small but committed minority that agrees with Edwards' contemporary
assessment that something important has been lost in turning away from a
more explicit, algorithmic standpoint. Although it is hard for mathematicians
with a modern training to recapture a nineteenth century algorithmic sensibility,
anyone studying the great works of that century cannot but appreciate the crisp
elegance and efficiency of the associated conceptions. (Edwards' expository
texts (1984, 1990, 1994, 1995, 1996, 2001, 2005) do a fine job of conveying a
feel for this style of thought.) The reader should therefore keep in mind that
in the discussion that follows, alternative, Kroneckerian points of view are not
adequately represented.

4   The first version of the theory of ideals

For both Dedekind and Kronecker, the first step towards extending Kummer's
theory to more general rings of integers involved finding the right definition of
an "integer." To see that this problem is not a trivial one, consider the primitive
cube root of 1 given by $\omega = (1 + \sqrt{-3})/2$. The field $\mathbb{Q}(\omega)$ obtained by adjoining

ω to the rational numbers coincides with the field $\mathbb{Q}(\sqrt{-3})$. What should we take the integers of that field to be? From the first representation, following Kummer's lead, we would take the integers to be those numbers of the form $x + y\omega$, where x and y are ordinary integers. From the second representation, we might take them to be the numbers of the form $x + y\sqrt{-3}$. These two choices are distinct; for example, ω itself is an integer on the first choice, but not on the second. The fact that two representations of the same field lead to different choices of integers may, in and of itself, raise concern. But the situation is even more serious: a simple argument shows that with the second choice, *no* good theory of ideal divisors is possible.[13]

The solution is to define the algebraic integers to be those complex numbers θ that are roots of monic polynomials with coefficients in $\mathbb{Z}$, that is, which satisfy an equation of the form

$$\theta^k + a_{k-1}\theta^{k-1} + \ldots + a_1\theta + a_0 = 0$$

for some choice of ordinary integers $a_0, a_1, \ldots, a_{k-1}$. Whereas there are various ways of motivating this choice of definition, Edwards notes (1980, Section 5) that the historical origin is unclear. In their presentations, Dedekind and Kronecker simply put it forth without further comment. In the example above, this definition sanctions the first choice as the correct one.

But even with an appropriate notion of "integer," the generalization of Kummer's theory posed a number of difficulties. In (1878), Dedekind reflects on the obstacles that had to be overcome:

> I first developed the new principles, through which I reached a rigorous and exceptionless theory of ideals, seven years ago, in the second edition of Dirichlet's *Lectures on Number Theory*, and more recently in the *Bulletin des sciences mathèmatiques et astronomiques*, presented in a more detailed and in a somewhat different form. Excited by Kummer's great discovery, I had previously worked for a number of years on this subject, though I based the work on a quite different foundation, namely, the theory of higher congruences; but although this research brought me very close to my goal, I could not decide to publish it because the theory obtained in this way principally suffers two imperfections. One is that the investigation of a domain of algebraic integers is initially based on the consideration of a definite number and the corresponding equation, which is treated as a congruence; and that the definition of ideal numbers (or rather, of divisibility by ideal numbers) so obtained does not allow one to recognize the *invariance* these concepts in fact have from the outset. The second imperfection of this kind of foundation is that sometimes peculiar exceptions arise which require special treatment. My newer theory, in contrast, is based exclusively on concepts like that of *field*,

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[13]See the footnotes to Sections §8 and §10 of (Dedekind 1877). The argument is spelled out in detail in (Edwards, 1980, Section 4).

integer, or *ideal*, that can be defined without any particular representation of numbers. Hereby, the first defect falls away; and just so, the power of these extremely simple concepts shows itself in that in the proofs of the general laws of divisibility no case distinction ever appears. (Dedekind, 1878, *Werke* vol. 1, 202–203; modified from an unpublished translation by Ken Manders and Dirk Schlimm. I have deleted Dedekind's references to page numbers and sections.)

Expanding on the mathematical developments described in this passage results in a "good news / bad news" story. A set $\{\omega_1, \omega_2, \ldots, \omega_k\}$ of integers in a finite extension of the rationals is said to be a *basis* for the ring of integers in that field when every element of the ring can be expressed uniquely as

$$a_1\omega_1 + a_2\omega_2 + \ldots + a_k\omega_k,$$

where $a_1, \ldots, a_{k-1}$ are ordinary integers. The good news is that when a ring of integers in a finite extension of $\mathbb{Q}$ has a basis of the form $\{1, \theta, \theta^2, \ldots, \theta^{k-1}\}$ for some element θ, Dedekind's theory of higher congruences provides a theory of ideal divisors generalizing Kummer's. The bad news is that not every such ring of integers has a basis of that form; in a sense, Kummer was lucky that this *is* the case with the cyclotomic integers. In general, one can always find algebraic integers θ such that $\{1, \theta, \theta^2, \ldots, \theta^{k-1}\}$ is a basis for the *field*, which is to say, the field is the set of linear combinations of those elements using *rational* coefficients. Furthermore, since every prime ideal divisor has to divide an ordinary integer p that is prime among the ordinary integers, it is enough to characterize the ideal prime divisors of each integer prime p. The good news is that the theory of higher congruences still works as long as one has a θ such that $\{1, \theta, \theta^2, \ldots, \theta^{k-1}\}$ is a basis for the field with the property p does not divide a certain integer value called the *discriminant* of this basis. The bad news is that when the discriminant is not ± 1, there will be finitely many primes p that *do* divide the discriminant, and so are not handled by the theory. These are the "peculiar exceptions" which "require special treatment." The good news is that one can still fashion a workable theory of ideals using a different choice θ for each integer prime p, so it suffices to have, for each p, a choice of θ such that p does not divide the discriminant. The bad news is that there are even cases where for a given p *no* choice of θ will work; Dedekind gives a specific example of a cubic extension of the rationals in which no choice of θ can be used to represent the ideal divisors of 2. The passage above ultimately concludes with a piece of good news, namely, that one can dispense with the theory of higher congruences entirely, and consider more general representations of ideal divisors. In this way, Dedekind happily reports, he has been able to obtain a theory which accounts for the ideal prime divisors, uniformly, and all at once.

Our discussion, to this point, has already highlighted a number of criteria that Dedekind takes to be important for a theory of ideal divisors. These include:

- Generality: the theory should apply to rings of integers beyond the ordinary integers, Gaussian integers, cyclotomic integers, and rings of quadratic integers that were useful to Euler.

- Uniformity: one theory should cover all these cases, and, indeed, one definition of the ideal divisors should account for all the ideal divisors in a given ring of integers. Furthermore, as much as possible, proofs should cover all situations uniformly, without case distinctions.

- Familiarity: the overall goal is to restore the property of unique factorization, which has proved to be important to the the ordinary integers, so that one can carry results forward to the new domains.

With respect to the last, Dedekind's writings show that he is acutely aware of the role that definitions play in structuring a theory. One finds him concerned with such issues of systematization as early as 1854, in his *Habilitationsrede*. There, he characterizes a process of extending operations like addition and multiplication to extended domains, whereby one identifies the laws they satisfy in a restricted domain, and stipulates that these laws are to maintain their general validity (see also the discussion in (Sieg and Schlimm, to appear)).

One reason for this requirement can be found in Dedekind's insistence that definitions and methods of proof used in an extended domain should parallel the definitions and methods of proof that have been effective in more restricted domains. For example, in his presentation of the theory of higher congruences, he highlights the goal of introducing concepts in such strong analogy to those of elementary number theory that only "a few words need to be changed in the number-theoretic proofs" (1857, *Werke* vol. 1, 40). In his presentations of ideal theory, he is careful to point out where definitions, basic characteristics, theorems, and proofs with respect to algebraic integers and ideals agree with their counterparts for the ordinary integers, and he seems to enjoy citing parallel developments in Dirichlet's *Lectures* wherever he can. The methodological benefits are clear, since it is often easy and efficient to reuse, adapt, and extend familiar modes of reasoning.

Thus far, we have seen little that would distinguish Dedekind's perspective from that of Kronecker, who is equally sensitive to the role of definitions in the structure of a theory, and who, for example, in his *Grundzüge* (1882), explicitly chose his basic concepts in such a way that they would remain unchanged when one passes from the rational numbers to algebraic extensions. From a methodological point of view, perhaps the most striking difference between Dedekind's theory and those of Kummer and Kronecker is Dedekind's use of the set-theoretic notion of an *ideal*. Recall that Kummer reasoned about his ideal divisors in terms of explicitly given predicates that express what it means for an algebraic integer x of the field in question to be divisible by the ideal divisor α. In contrast, Dedekind chose to identify the ideal divisor α with the set, or "system," $\mathfrak{a}$ of all the integers x that α divides. It is clear that this set is closed under addition, and under multiplication by any integer in the ring. Thus, Dedekind defined an ideal to be any system of elements of the ring of integers in question with these properties, and, later, showed that every such ideal arises from an ideal divisor in Kummer's sense.

Dedekind went out of his way to explain why this tack is to be preferred. Referring to the general lack of sufficient divisors in a ring of integers, Dedekind

writes that Kummer

> ...came upon the fortunate idea of nonetheless feigning [fingieren] such numbers μ' and introducing them as *ideal* numbers. The *divisibility* of a number α' by these ideal numbers μ' depends entirely on whether α' is a root of the congruence $\eta\alpha' \equiv 0 \pmod{\mu}$, and consequently these ideal numbers are only treated as moduli; so there are absolutely no problems with this manner of introducing them. The only misgiving is that the immediate transfer of the usual concepts of the *actual* numbers can, initially, easily evoke mistrust of the certainty of the proof. This has caused us to inquire after a means of clothing the theory in a different garb, so that we always consider *systems* of actual numbers. (Dedekind, 1871, *Werke* vol. 3, 221; my translation)

He returns to this point in 1877, explaining why Kummer's algorithmic treatment is not well-suited to the task at hand:

> We can indeed reach the proposed goal with all rigour; however, as we have remarked in the Introduction, the greatest circumspection is necessary to avoid being led to premature conclusions. In particular, the notion of *product* of arbitrary factors, actual or ideal, cannot be exactly defined without going into minute detail. Because of these difficulties, it has seemed desirable to replace the ideal number of Kummer, which is never defined in its own right, but only as a divisor of actual numbers ω in the domain $\mathfrak{o}$, by a *noun* for something which actually exists. (Dedekind, 1877, *Werke* vol. 3, 287; trans. Stillwell, 1996, 94)

Thus Dedekind maintains:

- The objects one refers to in a theoretical development (in this case, the ideal divisors) should be identified with *things*, not just used as modes of expression.

Indeed, Dedekind suggests that this is not just desirable, but even necessary when the mathematics is sufficiently complex.

In a sense, replacing the predicates P_α by the systems S_α of integers that satisfy them is mathematically inert. All the effects that mathematical objects can have on mathematical discourse are mediated by the roles they play in assertions; so if all references to ideal divisors are expressed in terms of the property of dividing an element of the ring, it makes little difference as to whether one takes "x has property P_α" or "x is an element of the set S_α" to stand duty for "α divides x." Both approaches require one to enforce the discipline that all properties of ideal divisors must ultimately be defined in terms of divisibility of integers, and Dedekind expresses concern that insufficient care may lead one astray (see also the discussion of multiplication of ideals in the next section). But, to be clear, this is a concern on either reading of the divisibility relation.

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The insistence on treating systems of numbers — viewed as either sets of numbers, or as predicates — as objects in their own right does, however, have important methodological consequences: it encourages one to speak of "arbitrary" systems, and allows one to define operations on them in terms of their behavior as sets or predicates, in a manner that is independent of the way in which they are represented. For example, in 1871, Dedekind defines the least common multiple of two modules to be their intersection, without worrying about how a basis for this intersection can be computed from bases for the initial modules. We find a similar use of nonconstructivity when he characterizes integral bases as those bases of integers whose discriminants have the least absolute value; he does this without giving an algorithm for finding such a basis or determining this least discriminant. In fact, in both examples just cited, algorithms can be obtained. But Dedekind's presentation sends the strong message that such algorithms are not *necessary*, i.e. that one can have a fully satisfactory theory that fails to provide them. This paves the way to more dramatic uses of nonconstructive reasoning, in which one uses facts about infinitary functions, sets, and sequences that are *false* on an algorithmic interpretation. Such reasoning was used, for example, by Hilbert, in proving his *Basissatz* in 1890.

Dedekind's insistence on having an explicit set-theoretic object to stand as the referent for a mathematical term is coupled with the following exhortation:

- The definition of new objects should be independent of the way they are represented, and arguments involving them should not depend on any particular representation.

That way, the arguments "explain" why calculations with and properties of the objects do not depend on these choices of representations. This is why Dedekind, in the passage above, complains that his first attempt at a theory of ideal divisors did not "allow one to recognize the invariance the concepts in fact have from the outset."

It is precisely these last two criteria that push Dedekind to the use of set-theoretic language and methods. In particular, his treatment commits him to acceptance of the following:

- Infinite systems of numbers can be treated as objects in their own right, and one can reason about the domain of *all* such objects.

This view is commonly accepted today, but it was novel to the mathematics of the 1870's.[14] It is striking that Dedekind adopts the use of such systems without

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[14] Authors like Laugwitz (1999) and Ferreirós (1999) take Riemann's general notions of manifold and function to be conducive to a set-theoretic viewpoint. It is not at all clear, however, how Riemann would have felt about Dedekind's use of definitions involving quantification over completed infinite totalities. There is a tentative discussion of this issue in (Ferreirós, 1999, Section II.4.2).

In 1883, Cantor found a precedent for the use of completed infinites in Bolzano's *Paradoxes of the infinite* of 1851. Surveying the philosophical literature to date, Cantor wrote that "Bolzano is perhaps the only one for whom the proper-infinite numbers are legitimate." (Cantor, 1883, §7 paragraph 7; trans. Ewald, 1996).

so much as a word of clarification or justification. That is, he simply introduced a style of reasoning that was to have decisive effects on future generations, without fanfare. When, in (1883), Cantor published an introduction to his theory of the infinite, he invoked considerations from history, philosophy and theology to assess the legitimacy of the use of completed infinite totalities, and responded to criticisms, both actual and anticipated. In (1888), Dedekind did spell out, informally, some of the rules he took to govern the use of infinite systems. But there he simply characterized such systems as "objects of our thought" and took this as sufficient justification of their legitimacy.[15]

5   The second and third versions of the theory of ideals

Dedekind's second version of the theory of ideals was published in 1876–1877. Since the next published version, which appeared in the third edition of the Dedekind-Dirichlet *Lectures*, is essentially a condensed version of that, I will focus on the second version here, and refer to it as the "1877 version."

There are at least two significant differences between the 1871 and 1877 versions of the theory. The first has to do with the treatment of multiplication of ideal divisors. In the presentation of 1871, after defining the notion of an ideal, Dedekind defines divisibility of an algebraic integer α by an ideal $\mathfrak{a}$ to mean that α is an element of $\mathfrak{a}$. He then defines the divisibility of one ideal by another, "$\mathfrak{b}$ divides $\mathfrak{a}$," to mean that every element of $\mathfrak{a}$ is an element of $\mathfrak{b}$. This makes sense when one considers that among the ordinary integers, the assertion that b divides a is equivalent to the assertion that every divisor of a is a divisor of b, and, indeed, this was the notion of divisibility used implicitly by Kummer. But it would be much more natural to say that $\mathfrak{b}$ divides $\mathfrak{a}$ whenever there is another ideal $\mathfrak{c}$ such that $\mathfrak{b}\mathfrak{c} = \mathfrak{a}$. At this stage in the development, however, Dedekind cannot do this, for the simple reason that he has not yet defined multiplication of ideals. In 1871, unique factorization is expressed by the fundamental theorem that every ideal is the least common multiple of all the powers of prime ideals that divide it, where the least common multiple of any finite set ideals is defined to be their intersection. Multiplication of ideals plays no role in the proof, and is, in fact, defined only afterwards.

In contrast, in 1877, multiplication of ideals is defined much earlier, and plays a central role in the presentation of the theory. Why the change? We have already noted Dedekind's sensitivity to the role that fundamental definitions play in structuring a theory. In modern terms, it is natural to express the goal of the theory of ideal divisors as being that of constructing a semigroup satisfying unique factorization, together with a suitable embedding of the integers of the field in question (up to associates). This is, for example, the characterization used in Weyl's Princeton lectures on algebraic number theory (1940), as well as

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[15]There is, of course, a good deal that can be said about how to interpret Dedekind's views in this respect; see, for example, (Ferreirós, 1999, Chapter VIII).

in more recent presentations, like Borevich and Shafarevich's textbook (1966). On this view, the goal is to define the collection of ideals *with an associated multiplication*, and to show that the resulting structure meets the specification. From that perspective, multiplication is naturally prior.[16]

One might object that one can equally well characterize the goal of a theory of divisors taking the notion of divisibility (and "prime power") as primitive. Kummer himself stated the requisite properties of the theory in such a way (see Edwards, 1980, Section 3), and Dedekind's 1871 version shows, directly, these these requirements are satisfied by the system of ideals. But this way of proceeding runs against Dedekind's insistence that definitions and methods of proof used in an extended domain should parallel the definitions and methods of proof that have been effective in more restricted domains, since, when it comes to the ordinary integers, divisibility is invariably defined in terms of multiplication.

Nonetheless, in 1877, Dedekind preserved the original 1871 notion of divisibility, making the theory of ideals diverge from the theory of integers almost immediately. The fact that there are two natural notions of divisibility at hand is confusing, but the good news is that, in the end, the two notions coincide. According to Dedekind's introduction to the 1877 version, this is something we come to see "only after we have vanquished the deep difficulties characteristic of the nature of the subject." Indeed, establishing the equivalence is tantamount to establishing unique factorization itself. To see this, note that unique factorization for the integers follows from the fact that the notions of "prime" and "irreducible" coincide. Passing to the theory of ideals, it is easy to show that every prime ideal is irreducible, and, further, that every prime ideal is prime with respect to the Dedekind-Kummer notion of divisibility. Demonstrating that the two notions of divisibility coincide therefore shows that there is sufficient agreement with the theory of the integers to ensure that unique factorization holds for ideals as well.

To sum up, in 1871, Dedekind did not define multiplication of ideals until the very end of the theoretical development, where it is clear that multiplication has all the expected properties. In contrast, in 1877, he defined multiplication much earlier in the development, at which point showing that multiplication has the expected properties becomes the central task. Dedekind felt that making this change allowed him to characterize the goal of the theory of ideal divisors in a way that highlights parallels with elementary number theory, and made it clear how one can attain these goals by resolving the apparent differences between the two. This highlights a number of benefits of being attentive to axiomatic properties:

- It makes it clear what properties of a particular domain one wants to preserve, when extending it to a more general domain;

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[16]Kronecker provided an entirely different, and elegant, formulation of the problem. On his approach, the central task is to find an appropriate definition of the *greatest common divisor* of any finite number of algebraic integers, whether it is actual or ideal. Edwards (1990) observes that there are many advantages to this way of setting up the theory. For example, whereas the factorization of an integer into ideal prime divisors changes as one expands the ring of integers in question, greatest common divisors remain invariant.

- It makes it clear what desired properties are *absent* from a particular domain, which one may wish to add.

- It can help suggest the definitions by which these two goals can be attained.

We will see in a moment that Dedekind compared his construction of the domain of ideal divisors of a ring of integers to his construction of the real numbers in (1872). Indeed, the two run parallel in a number of respects. In his construction of the reals, Dedekind is careful to isolate the order-theoretic properties of the rational numbers that will be preserved, and is concerned that arithmetic identities will carry over smoothly as well (see Dedekind, 1872, end of §6). He is also careful to identify the goal of the construction as satisfaction of the "principle of continuity," a property that clearly holds of the geometric line, but does not hold of the rationals. It is these considerations that suggest a suitable definition of the real numbers. Similarly, in his construction of ideal divisors, he is attentive to the properties of the ring of integers in a finite extension of the rationals, qua divisibility, that will be preserved; and the properties, familiar from the integers but generally absent from such rings of integers, which need to be added to obtain unique factorization. Again, these considerations suggest a suitable definition of the ideal divisors.

As far as the importance of defining multiplication from the outset, Dedekind describes the state of affairs as follows:

> Kummer did not define ideal numbers themselves, but only the divisibility of these numbers. If a number α has a certain property A, to the effect that α satisfies one more more congruences, he says that α is divisible by an ideal number corresponding to the property A. While this introduction of new numbers is entirely legitimate, it is nevertheless to be feared at first that the language which speaks of ideal numbers being determined by their products, presumably in analogy with the theory of rational numbers, may lead to hasty conclusions and incomplete proofs. And in fact this danger is not always completely avoided. On the other hand, a precise definition covering *all* the ideal numbers that may be introduced in a particular numerical domain $\mathfrak{o}$, and at the same time a general definition of their multiplication, seems all the more necessary since the ideal numbers do not actually exist in the numerical domain $\mathfrak{o}$. To satisfy these demands it will be necessary and sufficient to establish once and for all the common characteristic of the properties $A, B, C, \ldots$ that serve to introduce the ideal numbers, and to indicate, how one can derive, from properties A, B corresponding to particular ideal numbers, the property C corresponding to their product. (Dedekind, 1877, *Werke* vol. 3, 268–269; trans. Stillwell, 1996, 57)

Thus, Dedekind maintains that defining multiplication from the start makes it less likely that we will improperly transfer patterns of reasoning that we are

accustomed to use with the ordinary integers. The passage above also reiterates the value of uniformity, in the assertion that all the ideal numbers should be introduced by a single definition, all at once, and the relevant operation should be introduced at the same time. In a footnote to the passage above, he compares the situation to the development of the real numbers:

> The legitimacy, or rather the necessity, of such demands, which may always be imposed with the introduction or creating of new arithmetic elements, becomes more evident when compared with the introduction of *real irrational* numbers, which was the subject of a pamphlet of mine... Assuming that the arithmetic of *rational* numbers is soundly based, the question is how one should introduce the irrational numbers and define the operations of addition, subtraction, multiplication, and division on them. My first demand is that arithmetic remain free from intermixture with extraneous elements, and for this reason I reject the definition of real number as the ratio of two quantities of the same kind. On the contrary, the definition or creation of irrational number ought to be based on phenomena one can already define clearly *in the domain R* of rational numbers. Secondly, one should demand that all real irrational numbers be engendered simultaneously by a common definition, and not successively as roots of equations, as logarithms, etc. Thirdly, the definition should be of a kind which also permits a perfectly clear definition of the calculations (addition, etc.) one needs to make on the new numbers. (*ibid.*)

It is clear from the discussion that follows that by "definition of the calculations" in the last sentence Dedekind means set-theoretic definition, since he refers to his own (set-theoretic) definition of the product of two positive real numbers.

The claims made here represent a central part of Dedekind's world view. From a Kroneckerian perspective, introducing a new domain of "numbers" is a matter of introducing symbolic expressions, operations governing them, and methods of determining equality, or equivalence, of expressions. For example, if $f(x)$ is an irreducible polynomial with rational coefficients, one can extend the rational numbers, $\mathbb{Q}$, to a field with a root of $f(x)$, by considering the quotient $\mathbb{Q}[x]/(f(x))$. It is straightforward to extend the field operations to the larger domain and embed the original domain, $\mathbb{Q}$. Nonconstructively, it is easy to iterate this process, and obtain a field in which $f(x)$ factors into a product of linear polynomials: if $f(x)$ does not split into linear factors after the first adjunction of a root, pick any nonlinear factor of $f(x)$ that is irreducible in the new field, adjoin a root of that, and so on. It is a much greater challenge, however, to show that this can be done algorithmically. To start with, one needs a procedure to determine whether or not $f(x)$ splits into linear factors after the first step, and, if not, to find a nonlinear irreducible factor. In fact, Kronecker designed his "Fundamentalsatz der allgemeinen Arithmetik" (1887) to provide an algorithmic description of a splitting field for any given polynomial. He viewed this theorem as a proper formulation of the fundamental theorem

of algebra, since it establishes the "existence" of the polynomial's roots in a suitable extension of the rationals (see the discussion in (Edwards, 2005), as well as alternative constructions in (Edwards, 2005 and 1984)). Of course, one may later wish to deal with trigonometric functions, or the complex exponential or logarithm; here, again, the challenge is to give the appropriate rules for manipulating the relevant expressions and relate them to the prior ones. There is no further need to say what, exactly, the complex numbers *are*, to define them "all at once," or even to suppose that they exist; we simply expand our methods of representation and calculation as the need arises. (In the appendix to his calculus textbook (1994), Edwards illustrates this view with a "Parable of the mathematician and the carpenter.")

From Dedekind's point of view, this piecemeal approach will not do; this is exactly what the second criterion in the passage above is meant to rule out. Instead, we want definitions that determine *all* the real numbers, *all* the complex numbers, and *all* the functions we may wish to define on them, from the start. Of course, with Kronecker, we can go on to develop means of representing *particular* numbers. For a given choice representations, we can then ask how to calculate the values particular functions that are of interest. But a definition of the entire system should come first, and functions should be defined in a manner that is independent of any choice of representations. Issues having to do with calculation come later.

There is something to be said for such an approach, in that it serves to unify and guide the various extensions. A single uniform definition of the real numbers gives an account of what it is that particular expressions are supposed to represent, and a uniform definition of multiplication tells us, from the start, what it is that particular algorithms, based on particular representations, are supposed to compute. Verification that the algorithms behave as they are supposed to then guarantees that properties that have been shown to hold for multiplication, in general, also hold for the algorithmic instantiation. Of course, the downside is that one may generally define functions without knowing how to compute with representatives of elements in a given domain. Algorithmic issues, like the one suggested above, can be quite difficult, and one might credit the set-theoretic development for encouraging us to look for the appropriate algorithms (or to determine whether or not the functions are computable).[17]

There is evidence that Kronecker would have concurred with these assessments, to an extent. His student, Hensel, later explained that Kronecker was not dismissive of nonconstructive definitions and proofs; rather, he felt that a theory is not entirely satisfactory or complete until the algorithmic details are in place (see the introduction to Kronecker, 1901; the relevant excerpt is quoted in Stein, 1988). In other words, Kronecker may have recognized the value of nonconstructive methods in providing heuristic arguments, with the

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[17]For another example, note that the nonconstructive development of the reals makes it clear that if $f(x)$ is an odd-degree polynomial with rational coefficients, it has a real root; and if α is the least such root, one can extend not just the arithmetic operations on the rational numbers but also the ordering to the field $\mathbb{Q}(\alpha)$. Seeing that this can be done algorithmically requires a lot more work.

caveat that such arguments are not the subject's final goal, which is to have rigorous proofs. In contrast, Dedekind is willing to accept a nonconstructive argument as a rigorous proof in its own right.

Summarizing, then, we can ascribe to Dedekind the following views:

- We desire not just an axiomatic characterization of mathematical domains of interest, but constructions that yield *all* the (possible) elements, at once.

- Functions of interest should be defined from the start, uniformly, for all the elements of the domain.

- Nonconstructive, set-theoretic definitions are perfectly admissible, and nonconstructive proofs can be rigorous and correct.

- Issues regarding algorithms and explicit representations of elements come later in the development. Central properties of the domain in question should be established without reference to particular representations.

We have seen that in 1877 Dedekind provided a set-theoretic definition of the ideals in any ring of algebraic integers, and a set-theoretic definition of the product of two ideals. His goal was then to base the proof of unique factorization on these definitions, and to avoid relying on particular representations of the ideals. He was only partially successful in this respect, and, indeed, it is this concern that accounts for the second major difference between the 1871 and 1877 versions of the theory. In both 1871 and 1877, Dedekind defined a *prime ideal* to be an ideal whose only divisors are itself and the entire ring of algebraic integers. In both presentations, he showed that an ideal $\mathfrak{p}$ is prime if and only if whenever it divides a product $\alpha\beta$ of integers, it divides either α or β. In 1871, however, Dedekind went on to define a *simple ideal* to be a prime ideal that can be represented in a certain way, namely, as the set of all solutions π to a congruence $\nu\pi \equiv 0 \pmod{\mu}$. As was the case with Kummer's theory, this provides an effective test for divisibility by these ideal prime divisors: an algebraic integer π is divisible by the ideal divisor corresponding to μ and ν if and only if it satisfies the associated congruence. This can be extended to provide a test for divisibility by powers of these ideal divisors, cast in 1871 as a definition of the powers of the simple ideals. Dedekind showed that the notion of divisibility by powers of the simple ideals has the requisite properties; in particular, every element of the ring of integers is determined (up to associates) by the powers of the simple ideals that divide it. This implies that every prime ideal is a simple ideal, and that, in turn, implies that every ideal (other than the trivial ideal, $\{0\}$) can be represented by an appropriate μ and ν. Thus, every ideal in the new sense arises as the set of integers divisible by one of Kummer's ideal divisors. This is what Dedekind has in mind when he writes, in the 1877 introduction:

> A fact of highest importance, which I was able to prove rigorously only after numerous vain attempts, and after surmounting the greatest difficulties, is that, conversely, each system enjoying [the new definition of an ideal] is also an ideal [in Kummer's sense]. That is, it is

the set $\mathfrak{a}$ of all numbers α of the domain $\mathfrak{o}$ divisible by a particular number; either an actual number or an ideal number indispensable for the completion of the theory. (Dedekind, 1877, *Werke* vol. 3, 271; trans. Stillwell, 1996, 59–60)

Relying on simple ideals, however, runs counter to Dedekind's goal of avoiding reasoning that is based on particular representations of ideals rather than their "fundamental characteristics." By 1877, he has therefore dropped the term. That is not to say that he has avoided the use of such representations in his arguments: the ν and π above become κ and λ in a key argument in the 1877 version (§25), but they are deprived of the honorific status that is accorded by a definition, and the associated calculations are relegated to a pair of "auxiliary propositions" in the preceding section.

We will see in Section 6 below that Dedekind's inability to dispense with these calculations entirely remained a thorn in his side for years to come. In the meanwhile, his 1877 exposition and the mathematical context suggest a further reason that the calculations have been moved. Contemporary algebraic treatments of the theory of ideals tend to identify the most general classes of structures for which the various results of the theory hold; Dedekind's 1877 treatment is remarkably modern in this respect. Chapter 1 of that version (as well as §161 of the 1871 version) develops general theorems that are true of arbitrary modules.[18] In the 1877 version, he then, very self-consciously, develops the portion of the theory of ideals that only presupposes that one is dealing with a ring of integers whose rank as a module coincides with the degree of the extension. Following Dedekind, these structures are still called *orders* today. With a specific counterexample, Dedekind notes that not every order has a theory of ideal divisors (see footnote 13 above, and the preceding text), and then identifies the auxiliary propositions as being precisely the point at which one needs to assume that the ring in question is integrally closed, that is, consists of *all* the integers of the ambient number field. These propositions are clearly necessary for the ring to have a theory of ideal divisors; the subsequent development in 1877 shows that they are also sufficient. This is an instance of another methodological dictum that is in evidence in Dedekind's work:

- One should take great care to identify that axiomatic features of the domain in question that are in play at each stage of the development of a theory.

This ensures maximum generality, and also simplifies the theory by suppressing irrelevant distractions.

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[18]In both presentations, Dedekind defines a module to be a system of complex numbers that is closed under sums and differences. But at the end of Chapter 1 of (Dedekind, 1877), he notes that the "researches in this first chapter . . . do not cease to be true when the Greek letters denote not only numbers, but any objects of study, any two of which α, β produce a determinate third element $\gamma = \alpha + \beta$ of the same type, under a commutative and uniformly invertible operation (composition), taking the place of addition. The module $\mathfrak{a}$ becomes a *group* of elements. . . ." In other words, Dedekind observes that the results hold for any (torsion-free) abelian group, viewed (in modern terms) as a free module over $\mathbb{Z}$. Today we recognize that, in fact, they hold more generally for free modules over a principal ideal domain.

The observations regarding the necessity of considering *all* the integers of a field are also present in the 1871 version; the 1877 version simply makes them more prominent. It is easy to sympathize with Edwards, who feels that the resulting reorganization makes the proof of unique factorization seem ad-hoc and unmotivated. As noted above, Dedekind himself was never fully happy with this version of the proof. But in localizing and minimizing the role of representations and calculations, and making them secondary to structural systematization, Dedekind is exhibiting tendencies that have become hallmarks of modern mathematics.

6   The final version of the theory of ideals

In the supplements to the fourth edition of Dirichlet's *Lectures*, Dedekind published his final version of the theory of ideals (1894), yet again markedly distinct from his prior versions. In (1895), he also described an additional, intermediate version which he obtained in 1887, and which was later obtained, independently, by Hurwitz. Dedekind's goal in (1895) is to explain why he takes the 1894 version to be superior to this intermediate one. The mathematical details of these two versions are nicely summarized in (Edwards, 1980), and, unsurprisingly, Edwards' judgment of their relative merits is the opposite of Dedekind's. Dedekind's analysis is, as usual, rife with methodological claims, and it is well worth recounting some of them here.

Dedekind again finds the central focus of theoretic development to lie in the task of proving equivalence between the two notions of divisibility. He writes:

> In §172 of the third edition of the *Number Theory*, as well as in §23 of my essay *Sur la théorie des nombres entiers algébriques*, I have emphasized that the greatest difficulty to be overcome for the foundation of the theory of ideals lies in the proof of the following theorem:
>
> 1. If the ideal $\mathfrak{c}$ is divisible by the ideal $\mathfrak{a}$, then there is an ideal $\mathfrak{b}$ which satisfies the condition $\mathfrak{a}\mathfrak{b} = \mathfrak{c}$.
>
> This theorem, through which the relationship between the divisibility and multiplication of ideals is ascertained, is, in the presentation of the time, only provable at nearly the conclusion of the theory. This fact is palpable in a most oppressive way, especially since some of the most important theorems could be formulated in appropriate generality only gradually, by successively removing restrictive assumptions. I therefore came back often to this key point over the years, with the intention of obtaining a simple proof of Theorem 1, relating directly to the concept of the integers; or a proof of one of the following three theorems, which, as one easily sees, are of equal significance to the foundation of the theory:
>
> 2. Each ideal $\mathfrak{m}$ can, by multiplication with an ideal $\mathfrak{n}$, be turned into a principal ideal.

3. Every finite, nonzero module $\mathfrak{m}$, which consists of either integers or fractional algebraic numbers, can via multiplication by a module $\mathfrak{n}$, whose numbers are formed from those on $\mathfrak{m}$ in a rational way, be turned into a module $\mathfrak{mn}$, which contains the number 1 and consists only of integers.

4. From m algebraic numbers μ_r that do not all vanish, one can obtain, in a rational way, m numbers ν_s, which satisfy the equation

$$\mu_1 \nu_1 + \mu_2 \nu_2 + \ldots + \mu_m \nu_m = 1,$$

as well as the condition that all the m^2 products $\mu_r \nu_s$ are integers.

Now, these four theorems are completely equivalent, insofar as each of them can be obtained from each of the three remaining ones without difficulty. In such cases it often happens that one of the theorems, due to its simpler form, is more amenable to a direct proof than the others. In the previous example, clearly theorem 4, or also theorem 3, which differs only superficially in the use of the concept of a module, stands out as simpler than theorems 1 and 2, which deal with the more complicated concept of an ideal. (Dedekind, 1895, *Werke* vol. 2, 50–52; my translation. I have omitted two footnotes, and Dedekind's references to statements of the theorems in his previously published works.)

Even in this short excerpt, we can discern a number of general claims, including the following:

- Theorems should be stated at an "appropriate level of generality," even though it is often not evident how to do this at the outset.

- An important theorem should have a proof which "relates directly" to the relevant concepts.

- Sometimes, casting a theorem in a "simpler" or more general form makes it easier to find a direct proof.

These can be viewed as calls for a kind of methodological directness, or purity. In the case at hand, even though Theorems 1–4 above are easily shown to be equivalent, Dedekind takes Theorems 3 and 4 to be preferred, because they deal with the more general concept of a module rather than an ideal. (A module, in Dedekind's terminology, is what we would call a $\mathbb{Z}$-module, and has only an additive structure; the concept of an ideal relies on the notion of multiplication as well.)

Once again, a summary of the mathematical developments reported on in (Dedekind, 1895) will help us grasp the methodological import. In 1882, Considering Kronecker's great work on algebraic number theory (1882), Dedekind found what he took to be a gap in a proof of a fundamental theorem. But he was able to prove a special case, a generalization of Gauss's theorem on the

product of primitive polynomials, which he later published and referred to as his "Prague Theorem." Furthermore, the Prague Theorem implies Theorem 3 above. This was also the tack discovered by Hurwitz, who took the Prague Theorem to be implicit in a work of Kronecker from 1883. But Dedekind felt that the detour through properties of polynomials with coefficients in the ring of integers destroys the theory's uniform character ("einheitlichen Charakter"), and is therefore unacceptable. After the commenting on the quote from Gauss that we discussed in Section 3 above, Dedekind writes:

> As a result, one will understand that I preferred my definition of an ideal, based on a characteristic inner property, to that that based on an external form of representation, which Mr. *Hurwitz* uses in his treatise. For the same reasons, I could not be fully satisfied with the proof of Theorem 3 mentioned above, based on [the Prague Theorem], since, by mixing in functions of variables the purity of the theory is, in my opinion, tarnished... (*ibid.*, 55)

So, Dedekind went back to the drawing board. Considering a very special case of the Prague Theorem, he reproduced a proof of a special case of Theorem 4 that he had discovered earlier, which works only for modules generated by two elements. But now, with the Prague Theorem under his belt, he saw how to extend the argument inductively to a full proof of Theorems 3 and 4. That is, he found a direct proof of one of the "simpler" statements that he was after, one that avoids passage through areas he took to be external to the theory. But even this was not enough to satisfy him:

> Hereby I had finally found what I had long sought, namely, a truly appropriate proof of Theorems 3 and 4, and therefore also the foundation for the new formulation of my theory of ideals. However, I was not entirely satisfied with this inductive proof, since it is dominated by mechanical calculation... (*ibid.*, 57)

With some additional reflection, Dedekind discovered a general identity involving modules,

$$(\mathfrak{a} + \mathfrak{b} + \mathfrak{c})(\mathfrak{bc} + \mathfrak{ca} + \mathfrak{ab}) = (\mathfrak{b} + \mathfrak{c})(\mathfrak{c} + \mathfrak{a})(\mathfrak{a} + \mathfrak{b}),$$

which forms the basis for a proof of Theorem 3 by induction on the number of generators of the module $\mathfrak{m}$. It is this proof which finally met his approval, and which appears in (1894).

Dedekind's narration is remarkable. We find him, true to form, doggedly determined to eliminate (or hide) any trace of calculations from his proofs, to present his proofs in such a way that they do not rely on any extraneous properties of the structures at hand, and to base his proofs only on the "intrinsic" properties of these structures rather than particular representations of the elements. It is exactly these features that Emmy Noether praised in her editorial notes in Dedekind's *Werke*, and exactly these features that Edwards finds artificial and disappointing. In the decades that followed, the attitudes

that Dedekind expresses here were to have dramatic effects on the course of mathematics, and so it is well worth trying to understand what lies behind them.

7   Coming to terms with methodology

We have considered a number of reasons that Dedekind felt that later versions of his theory of ideal divisors were successive improvements over prior ones, as well as over Kummer's, Hurwitz's, and Kronecker's theories. Probing these judgments and submitting them to careful scrutiny can help us get a better handle on the epistemological goals that drove Dedekind, as well as the benefits later generations saw in the methods he introduced. This illustrates a way in which it is possible to use of the history of mathematics to develop a robust epistemology, one that can account for the broad array of value judgments that are employed in the practice of mathematics. Put simply, the strategy is as follows: find an important mathematical development, and then explain, in clear terms, what has changed. Understanding how these changes are valued, or inquiring as to why these changes are valued, can yield insight as to what is fundamentally important in mathematics.

But, on closer inspection, the appropriate means of proceeding is not so clear. To describe "what has changed," one needs a characterization of the local state of affairs "before" and "after." And that raises the question of how the relevant states of affairs are to be described, that is, what features we take to be essential to characterizing a state of mathematical knowledge (or, perhaps, understanding, or practice) at a given point in time. But this, of course, presupposes at least some aspects of a theory of mathematical knowledge. So we are right back where we started: any use of historical case studies to develop the philosophy of mathematics will necessarily be biased by philosophical presuppositions and the very terms used to describe the developments (as well as, of course, a bias as to whether these developments constitute "advances"). Lakatos describes this state of affairs in the philosophy of science more generally:

> In writing a historical case study, one should, I think, adopt the following procedure: (1) one gives a rational reconstruction; (2) one tries to compare this rational reconstruction with actual history and to criticize both one's rational reconstruction for lack of historicity and the actual history for lack of rationality. Thus any historical study must be preceded by a heuristic study: history of science without philosophy of science is blind. (Lakatos, 1976, 138. The entire passage is parenthetic and italicized in the original.)

This give-and-take is unavoidable: the best we can do is present a theory of mathematical knowledge and see how it fares with respect to mathematical, logical, historical, and philosophical data; and then use the results of this evaluation to improve the philosophical theory.

My goal here is not to develop such a theory. In (Avigad, to appear), I suggest that a syntactic, quasi-algorithmic approach should be fruitful. That is, I expect that it will be useful to characterize states of knowledge not only in terms of collections of definitions, theorems, conjectures, problems, and so on, but also in terms of the available methods for dealing with these; e.g. methods of applying definitions, verifying inferences, searching for proofs, attacking problems, and forming conjectures. But rather than extend this speculation now, I would like to consider the ways that our discussion of Dedekind's work might inform, and be informed by, such a theory.

I have noted in the introduction that philosophical discussions of Dedekind's use of axiomatic methods usually have a metaphysical character, whereby the goal is to square the use of such reasoning with an appropriate "structuralist" view of mathematical objects. But we can also try to understand these axiomatic methods in terms of their impact on mathematical activity, and their benefits and drawbacks. We have come across a number of possible benefits in our analysis:

1. Axiomatization allows one to state results in greater generality.

2. It suggests appropriate generalizations.

3. It suggests appropriate definitions.

4. It allows one to transfer prior results smoothly, or adapt prior proofs to a new setting.

5. It simplifies presentations by removing irrelevant distractions, allowing one to focus on the "relevant" features of the domain at hand at each stage of a development.

A good philosophical theory of mathematics should help us make sense of these claims.

Similarly, rather than focus on metaphysical justifications for a set-theoretic ontology, we can try to come to terms with some of the mathematical influences of set-theoretic language and methods:

1. They allow us to treat predicates, and properties, as objects in their own right.

2. They allow us to provide uniform definitions of mathematical domains, and to refer to "arbitrary" elements of those domains and "arbitrary" functions on those domains.

3. They allow us to obtain results that hold generally, and apply to particular elements and functions that may be introduced in the future, with proofs that are independent of the manner of representing these elements and functions.

4. They can suppress tedious calculational information.

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We have seen that these effects are far from cosmetic, and generally force us to relinquish an algorithmic understanding of mathematical assertions and proofs. In other words, the use of set-theoretic methods comes at a serious cost, since these methods ignore or distract our attention from algorithmic issues that are also of great mathematical importance. A good philosophical analysis should put us in a better position to weigh the virtues against the vices.

And what are we to make of Dedekind's general aim of providing a "conceptual" approach to mathematics? In trying to map out a hierarchy of values and goals in mathematics, one option is to take this to be a goal in its own right: every well-founded hierarchy has to bottom out somewhere. An alternative is to *identify* Dedekind's conceptual approach with the use of axiomatic and set-theoretic methods, which are to be justified by appeal to different mathematical goals, such as simplicity, uniformity, or generality. One might even, paradoxically, try to justify a conceptual approach by its ability to support algorithmic developments, by providing a rigorous framework that can guide the search for effective representations and algorithms. Or one might, instead, try to justify a conceptual approach on pragmatic grounds, in terms of ease of learnability and communicability; on more general esthetic grounds; or on external grounds, like social utility or applicability to the sciences.

In trying to sort this out, it would be a mistake, I think, to expect a simple narrative to provide a satisfactory theory. I believe it is also a mistake to try to fashion such a theory from the top down, before we have begun to make some sense of the basic data. What we need first is a better philosophical analysis of the various value-laden terms that I have just bandied about, and the way they play out in particular mathematical settings. There is enough going on in the development of the theory of ideals, in particular, to keep us busy for some time.

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