Computability and Efficiency in Learning
An Ockham's Razor Account

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Outline

Epistemic Spaces

Learning and Learnability

Computational Assumptions

Ockham's Razor & Conclusive Learning

Ockham's Razor & Limiting Learning

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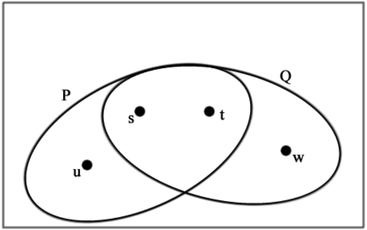
Ockham's Razor & Limiting Learning

An agent's uncertainty is represented by an epistemic space (S, Φ), where:

- $S = \{s_0, s_1, \ldots\}$ of epistemic possibilities, or possible worlds, and
- $\Phi \subseteq \mathcal{P}(S)$ a family of propositions.

Φ represent facts or observables being true or false in possible worlds.



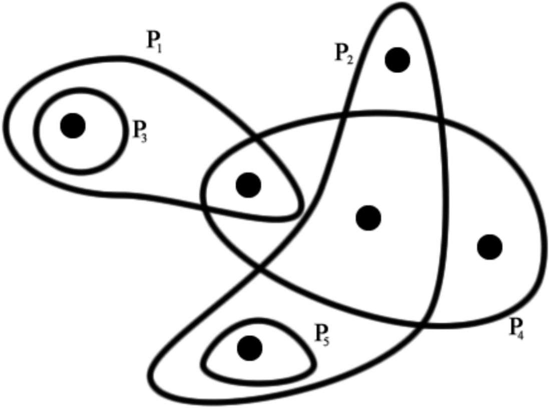
success of learning $\sim$ converging to the truth

Learning Power

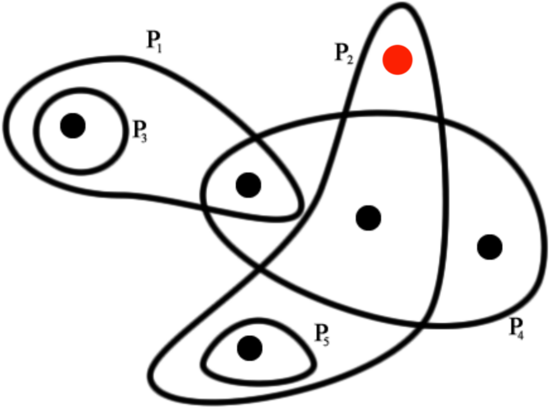
success of learning $\sim$ converging to the truth

- with certainty
- in the limit
- gradually
- ...

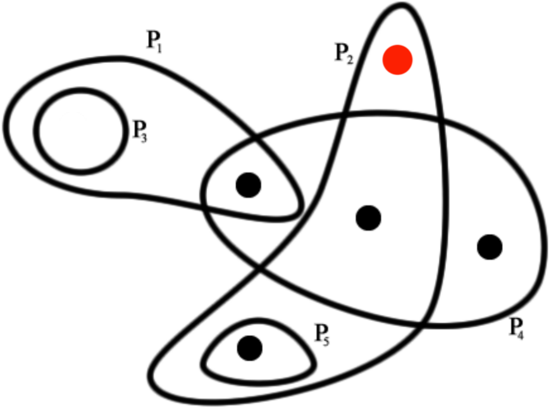
Learning Aims at Resolving Uncertainty



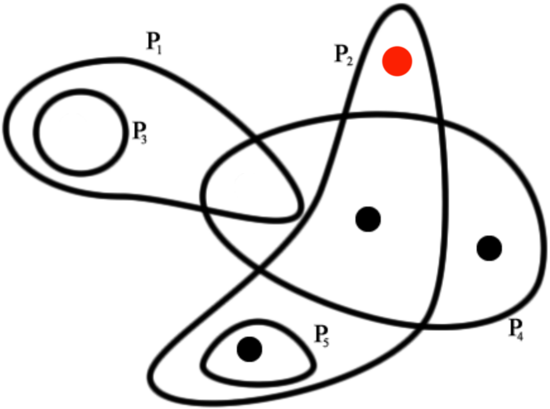
Learning via Update



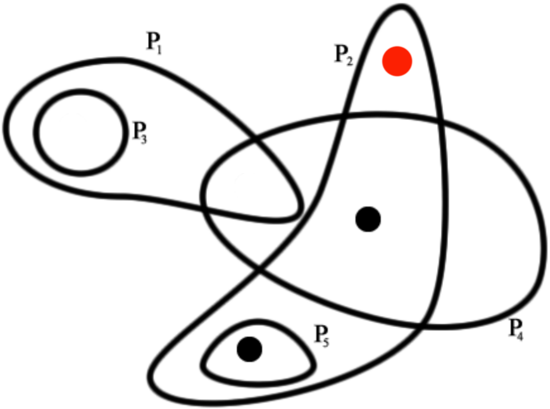
Learning via Update



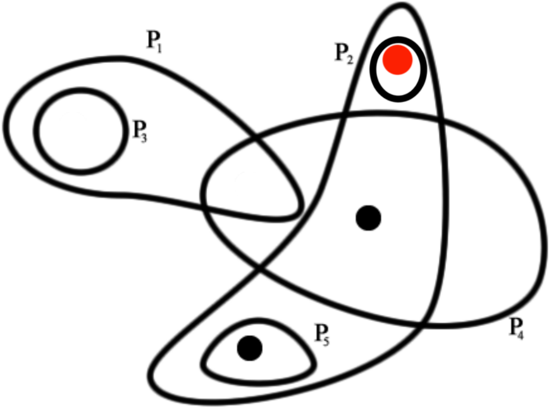
Learning via Update



Learning via Update: and what now?



Learning via Update: Towards Certainty



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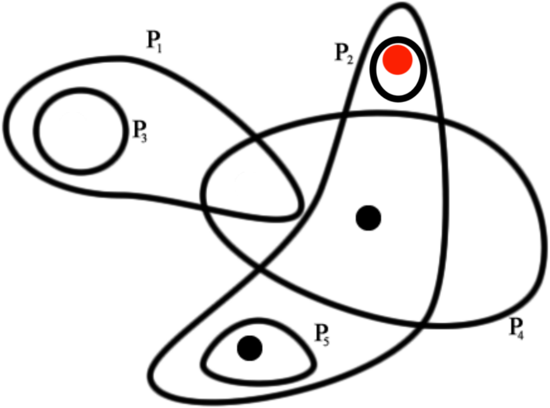

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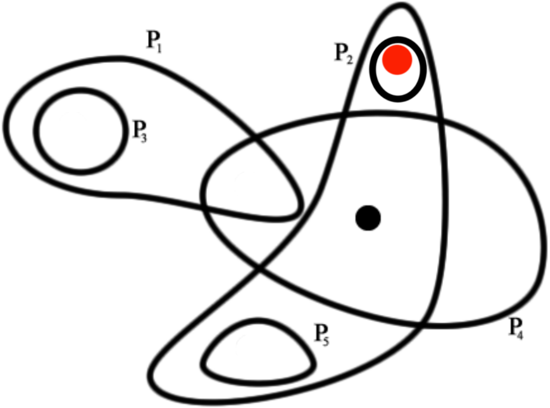

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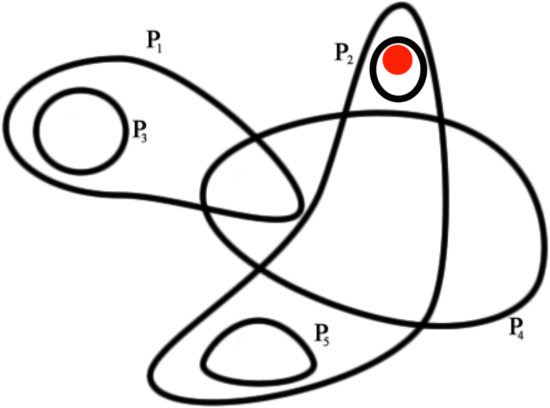
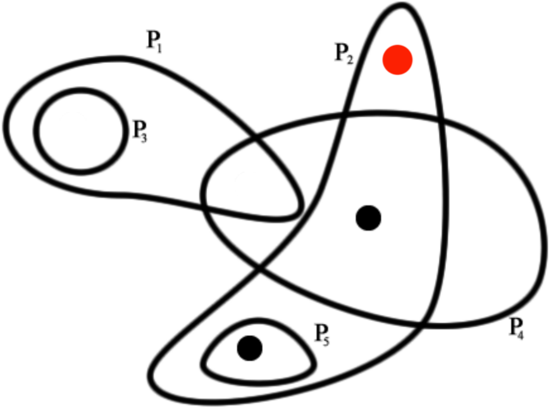
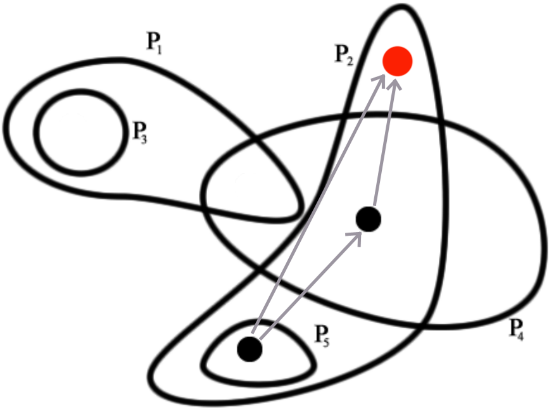



P1
P2
P3
P4
P5

Learning via Update: Towards Stable True Belief



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Ockham's Razor & Conclusive Learning

Ockham's Razor & Limiting Learning

- Learner L receives information about a possible world (the actual one).
- The information is an open-ended (infinite) sequence of propositions.
- Data stream $\varepsilon = (\varepsilon_1, \varepsilon_2 \ldots)$ is a data stream for $s \in S$ just in case

$$\{\varepsilon_n \mid n \in \mathbb{N}\} = \{p \in \Phi : s \in p\}.$$

- We write $\varepsilon \upharpoonright n$ for the sequence $(\varepsilon_1, \ldots, \varepsilon_n)$.
- Learner L is a function that on input of an epistemic space (S, Φ) and a finite sequence of observations $\sigma = (\sigma_0, \ldots, \sigma_n)$ outputs a hypothesis, i.e.,

$$L((S, \Phi), \sigma) \subseteq S.$$

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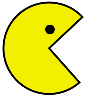

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Definition
Learner $L : \mathbb{N}^* \to \mathbb{N}$ is a computable function.

Definition
An epistemic space (S, Φ), $S = \{s_0, s_1, s_2, \ldots\}$, and $\Phi = \{p_0, p_1, p_2, \ldots\}$, is uniformly decidable just in case there is a computable function $f : S \times \Phi \to \{0, 1\}$ such that:

$$f(s, p) = \begin{cases} 1 & \text{if } s \in p, \\ 0 & \text{if } s \notin p. \end{cases}$$

- In epistemic logic uniform decidability is is guaranteed by finiteness.
- However the problem is non-trivial, e.g., in scientific scenarios.
- Epistemic space represents the uncertainty of a TM-representable mind.
- Subjective perspective on problem posing.
- Simple and appealing condition vs properties of convergence to knowledge.

(S, Φ)
$s_1 : p_1, p_3, p_4$
$s_2 : p_2, p_4, p_5$
$s_3 : p_1, p_3, p_5$
$s_4 : p_4, p_6$

(S', Φ')
$t_1 : p_1, p_3, p_4$
$t_2 : p_2, p_4, p_5$
$t_3 : p_1, p_3, p_5$
$t_4 : p_1, p_3, p_4, p_6$

(S, Φ)
s_1 : 1, 3, 4
s_2 : 2, 4, 5
s_3 : 1, 3, 5
s_4 : 4, 6

(S', Φ')
t_1 : 1, 3, 4
t_2 : 2, 4, 5
t_3 : 1, 3, 5
t_4 : 1, 3, 4, 6

Some Types of Learnability

(S, Φ)
s_1 : 1, 3, 4
s_2 : 2, 4, 5
s_3 : 1, 3, 5
s_4 : 4, 6

(S', Φ')
t_1 : 1, 3, 4
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t_3 : 1, 3, 5
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(S, Φ)
$s_1 : 1, 3, 4$
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$s_3 : 1, 3, 5$
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Conclusive Learnability

- Certainty in finite time.
- Only one answer,
- based on certainty.
- No chance to change later.

Some Types of Learnability

(S, Φ)
s_1 : 1, 3, 4
s_2 : 2, 4, 5
s_3 : 1, 3, 5
s_4 : 4, 6

(S', Φ')
t_1 : 1, 3, 4
t_2 : 2, 4, 5
t_3 : 1, 3, 5
t_4 : 1, 3, 4, 6

Limiting Learnability

- No certainty.
- Sequence of answers,
- based on reliability.
- Always a chance to change.

(S', Φ')
t_1 : 1, 3, 4
t_2 : 2, 4, 5
t_3 : 1, 3, 5
t_4 : 1, 3, 4, 6

Conclusive vs Limiting Learning

Limiting Learnability

- No certainty.
- Sequence of answers,
- based on reliability.
- Always a chance to change.

Conclusive Learnability

- Certainty in finite time.
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Search for a notion of simplicity that would guarantee that

> *always choosing the simplest theory compatible with experience and hanging on to it while it remains the simplest is both necessary and sufficient for efficiency of inquiry*

Learning and Ockham's Razor: ASAP

Search for a notion of simplicity that would guarantee that

> *always choosing the simplest theory compatible with experience and hanging on to it while it remains the simplest is both necessary and sufficient for efficiency of inquiry*

Efficient Inquiry $\rightarrow$ Efficient Conjecturing $\rightarrow$ Solution A.S.A.P.

- Conclusive Learning $\rightarrow$ Fastest Learning
- Limiting Learning $\rightarrow$ Conservative Learning

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The range of learning function L is extended by ? ("I do not know").

Definition

Learning function L is once defined on (S, Φ) iff for any stream ε for any world in S there is exactly one $n \in \mathbb{N}$ such that $L(\varepsilon \restriction n)$ is not an ?-answer.

Definition

Take an epistemic space (S, Φ).

- A world $s_m \in S$ is conclusively learnable in a computable way by a function L if L is computable, once-defined, and for every data stream ε for s_m, there exists a finite stage k such that $L((S, \Phi), \varepsilon_0, \ldots, \varepsilon_k) = \{s_m\}$.

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- The epistemic space (S, Φ) is said to be conclusively learnable in an computable way by L if L is computable and all its worlds in S are conclusively learnable in an computable way by L.
- Finally, the epistemic space (S, Φ) is conclusively learnable in an computable way just in case there is a computable learning function that can conclusively learn it.

DEFINITION

Take (S, Φ). A set $D_i \subseteq \Phi$ is a definite finite tell-tale set (DFTT) for s_i in S if:

1. D_i is finite,
2. $s_i \in \bigcap D_i$, and
3. for any $s_j \in S$, if $s_j \in \bigcap D_i$ then $s_i = s_j$.

THEOREM (MUKOUCHI 82, LANGE & ZEUGMANN 82)

(S, Φ) *is conclusively learnable in an computable way just in case there is a computable function* $f : S \to \mathcal{P}^{<\omega}(\Phi)$ *s.t.* $f(s)$ *is a DFTT for* s*.*

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Theorem (Mukouchi 82, Lange & Zeugmann 82)

(S, Φ) is conclusively learnable in an computable way just in case there is a computable function $f : S \to \mathcal{P}^{<\omega}(\Phi)$ s.t. $f(s)$ is a DFTT for s.

a world is conclusively learnable

if it makes true a finite conjunction of propositions

that together is false everywhere else

DEFINITION

Take (S, Φ) and $x \in \Phi$. The eliminative power of x with respect to (S, Φ) is determined by a function $El_{(S,\Phi)} : \Phi \to \mathcal{P}(\mathbb{N})$, such that:

$$El_{(S,\Phi)}(x) = \{i \,|\, s_i \notin x \;\&\; s_i \text{ in } S\}.$$

Additionally, for $X \subseteq \Phi$ we write $El_{(S,\Phi)}(X)$ for $\bigcup_{x \in X} El_{(S,\Phi)}(x)$.

eliminative power of a proposition is the complement of its extension

Definition (Fin-Id Problem)

Instance: A finite epistemic space (S, Φ), a world s_i in S.
Question: Is s_i conclusively learnable within (S, Φ)?

Theorem

Fin-Id *Problem is in P.*

Minimality of DFTT's: Two Kinds

set	a minimal DFTT	minimal-size DFTTs
$\{5, 7, 8\}$	$\{7, 8\}$	$\{5, 8\}$ or $\{7, 8\}$
$\{6, 8, 9\}$	$\{8, 9\}$	$\{6\}$
$\{5, 7, 9\}$	$\{7, 9\}$	$\{5, 9\}$ or $\{7, 9\}$
$\{8, 10\}$	$\{10\}$	$\{10\}$

Minimality of DFTT's: Complexity

finding a minimal DFTT is easy

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PROPOSITION

Let (S, Φ) be a conclusively learnable finite epistemic space. Finding a minimal DFTT of s_i in (S, Φ) can be done in polynomial time w.r.t. $card(\{x | s_i \in x\})$.

Minimality of DFTT's: complexity

finding a minimal-size DFTT is (most probably) harder

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Definition (Min-size DFTT Problem)

Instance: (S, Φ), $s_i \in S$, and $k \leq card(\{p|s_i \in p\})$.
Question: Is there a DFTT X_i of s_i of size $\leq k$?

Theorem

The Min-size DFTT *Problem is NP-complete.*

finding a minimal-size DFTT is (most probably) harder

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teaching efficiently might be hard

Preset Learning

Learners taking a more prescribed course of action by basing their conjectures on symptoms (DFTTs).

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Objection: infinite collections of DFTTs.
Solution: f_{dftt}, which for a finite X and s_i says if X is a DFTT of s_i.

If (S, Φ) is conclusively learnable
then there is f_{dftt} that for each world recognizes at least one DFTT.

Preset Learning: Some Theorems

1. conclusive learnability = preset conclusive learnability
2. preset learners are exactly those that react solely to the content

Fastest learner:

conclusively learns a world s_i as soon as objective 'ambiguity' disappears;

settles on the right world as soon as any DFTT for it has been given.

Definition

(S, Φ) is *conclusively learnable in the fastest way* if and only if there is a learning function L such that, for each ε and for each $i \in \mathbb{N}$,

$$L(\varepsilon \restriction n) = i \quad \text{iff} \quad \exists D_i^j \in \mathbb{D}_i \; (D_i^j \subseteq \mathrm{set}(\varepsilon \restriction n)) \; \& \; \neg \exists D_i^k \in \mathbb{D}_i \; (D_i^k \subseteq \mathrm{set}(\varepsilon \restriction n-1)).$$

Such L is a *fastest learning function*.

Theorem

There is a uniformly decidable epistemic space that is conclusively learnable, but is not conclusively learnable in the fastest way.

fastest conclusive learnability is properly included in conclusive learnability

Definition (Smullyan 1958)

Let $A, B \subset \mathbb{N}$. A separating set is $C \subset \mathbb{N}$ such that $A \subset C$ and $B \cap C = \emptyset$. In particular, if A and B are disjoint then A itself is a separating set for the pair, as is B. If a pair of disjoint sets A and B has no computable separating set, then the two sets are computably inseparable.

Let A and B be two disjoint r.e. computablely inseparable sets, such that:

- $x \in A$ iff $\exists y\, Rxy$ with R computable, and
- $x \in B$ iff $\exists y\, Sxy$ with S computable.

For each x there is at most one y, s.t. Rxy and at most one y, s.t. Sxy.
We define $(S_i)_{i \in \mathbb{N}}$:

$$S_i = \{2i, 2i+1\} \cup \{2j \mid Rji\} \cup \{2j+1 \mid Sji\}.$$

Proof: Illustration

$$
\begin{array}{l}
S_0 : p_0, p_1 \\
S_1 : p_2, p_3 \\
S_2 : p_4, p_5 \\
S_3 : p_6, p_7 \\
\vdots \\
S_i : p_{2i}, p_{2i+1} \\
\vdots \qquad\qquad ?
\end{array}
$$

Proof, part 2

The idea is that $S_i = \{2i, 2i + 1\}$ except that, for some m, Rim or Sim may be true, and then $2i \in S_m$ or $2i + 1 \in S_m$, respectively.

Note that:

- There can be at most one such m, and for that m only one of Rim or Sim can be true.
- Since A and B are computably inseparable there is no computable f that makes the choice for each i.
- Except for such intruders the languages are disjoint.

Proof, part 3

The argument:

- $\{2i, 2i+1\}$ is a DFTT for S_i.
- But, $\{2i+1\}$ is a DFTT for S_i if $i \notin B$, and $\{2i\}$ is a DFTT for S_i if $i \notin A$.
- However, a computable function that would give the minimal DFTTs of S_i gives a computable separating set of A and B.
- And this is impossible, since A and B are computably inseparable.

So there cannot be a computable fastest learner!

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Limiting Learnability: Definition

Definition

Take an epistemic space (S, Φ).

- A world $s_m \in S$ is limiting learnable in a computable way by a function L if L is computable, and for every data stream ε for s_m, there exists a finite stage n such that for all $k > n$, $L((S, \Phi), \varepsilon_0, \ldots, \varepsilon_k) = \{s_m\}$.

DEFINITION

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- The epistemic space (S, Φ) is said to be limiting learnable in an computable way by L if L is computable and all its worlds in S are limiting learnable in an computable way by L.
- Finally, the epistemic space (S, Φ) is limiting learnable in an computable way just in case there is a computable learning function that can limiting learn it.

DEFINITION

A learner L is conservative if, for each sequence σ and x

$$L(\sigma) \in \bigcap content(\sigma^\wedge\langle x\rangle) \text{ implies } L(\sigma^\wedge\langle x\rangle) = L(\sigma).$$

THEOREM

There is a uniformly decidable (S, Φ) *that is computably limiting learnable, but not by a computable conservative learner.*

Restrictiveness of Conservativity: Proof, part 1

0	φ_0	$s_0 = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \ldots\}$	$s'_0 = \{\langle 0,1\rangle\}$ or $\{\langle 0,1\rangle, \ldots \langle 0,n\rangle\}$
1	φ_1	$s_1 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \ldots\}$	$s'_1 = \{\langle 1,1\rangle\}$ or $\{\langle 1,1\rangle, \ldots \langle 1,n\rangle\}$
2	φ_2	$s_2 = \{\langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \ldots\}$	$s'_2 = \{\langle 2,1\rangle\}$ or $\{\langle 2,1\rangle, \ldots \langle 2,n\rangle\}$
3	φ_3	$s_3 = \{\langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \ldots\}$	$s'_3 = \{\langle 3,1\rangle\}$ or $\{\langle 3,1\rangle, \ldots \langle 3,n\rangle\}$
...	...	...	...

Take $j \in \mathbb{N}$, $s_j \in S$, and φ_j. Think of φ_j as of a (not necessarily successful) learner. Take a text for s_j, $t^j = \langle j,0\rangle, \langle j,1\rangle, \langle j,2\rangle, \langle j,3\rangle, \ldots$ If φ_j happens to identify s_j, then on some $t^j{\restriction}n+1$, φ_j will output j (obviously, if φ_j does not identify s_j, this does not have to happen).

$$s'_j = \begin{cases} \{\langle j,0\rangle, \ldots, \langle j,n\rangle\} & \text{where } \langle n,k\rangle \text{ are the smallest s.t.} \\ & \{\langle j,0\rangle, \ldots, \langle j,n\rangle\} \subset W_{\varphi_j^k(t^j{\restriction}n+1),k}; \\ \{\langle j,0\rangle\} & \text{if such a pair does not exist.} \end{cases}$$

Assume, towards contradiction, that conservative learner L learns (S, Φ) in the limit. L is in fact φ_j for some $j \in \mathbb{N}$ and it identifies $s_j \in S$. Take $t^j = \langle j,0\rangle, \langle j,1\rangle, \langle j,2\rangle, \langle j,3\rangle, \ldots$, then there will be $s'_j = \{\langle j,0\rangle, \ldots, \langle j,n\rangle\}$ in S. Take the text $\langle j,0\rangle, \ldots, \langle j,n\rangle, \langle j,n\rangle, \langle j,n\rangle, \ldots$ for s'_j. On the first occurrence of $\langle j,n\rangle$, φ_j will output i for s_j, and since the rest does not contradict s_j, φ_j will not retract (because it is conservative). Hence, φ_j will not identify s'_j. Contradiction.

It remains to be shown that (S, Φ) is limit learnable by a computable L. Depending on the first element seen by L:

1. $\langle j, m\rangle$, with $m \neq 0$, then L will output an index of s_j on any sequence σ extending $\langle j, m\rangle$, unless it is the case that $\{\langle j, 0\rangle, \ldots, \langle j, n\rangle\} \subset W_{\varphi_j^k(t^j \upharpoonright n+1), k}$ for some $\langle n, k\rangle \leq lh(\sigma)$. If it is so, it can be determined if all elements of σ are members of s'_j (since both σ and s'_j are finite). If that is the case L outputs s'_j and continues doing so as long as all the elements of the input sequence are elements of s'_j. If that is not the case L switches back to s_j.
2. $\langle j, 0\rangle$, then L conjectures s'_j as long as $\langle j, 0\rangle$ is the only pair seen, otherwise L switches to s_j and continues according to the behavior described before.

Conclusions

- Complexity of learning/teaching strategies in conclusive learning.
- Complexity of min-DFTT and min-size DFTT related concepts.
- The notion of preset learner in conclusive learning.
- Fastest learning is restrictive wrt to conclusive learnability.
- Conservative learning is restrictive wrt to limiting learnability

  even if computable convergence to certainty or safe belief is possible

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even if computable convergence to certainty or safe belief is possible
it may not be computably reachable just when objective ambiguity disappears
or when the learner is conservative in his mind changes

BR intuitive, determinate manners of updating models

FLT no prescribed ways of learning but often restricted by computability

Compare the two aspects: determinateness and computability.

Thank you!

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