Online Technical Appendix for
"The Emergence of Opinion Leaders in a Networked Online Community: A Dyadic Model with Time Dynamics and a Heuristic for Fast Estimation"

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Online Technical Appendix A

Performance of the Weighted Exogenous Sampling with Bayesian Inference (WESBI) Method

In this technical appendix, we examine the efficacy of the Weighted Exogenous Sampling with Bayesian Inference (WESBI) method for estimating a proportional hazard network growth model. We conduct a comprehensive simulation study covering a large variety of possible network structures characterized by different parameter values. For each network structure, we show that by sampling a small proportion of the total observations, we can recover the true network generating parameters with very high accuracy.

The basic simulation process for each of the sets of parameter values we use is the following: First, we simulate a network according to the parameter values in the set. Second, we consider different sampling proportions of this simulated network; for each sampling proportion, we sample the simulated network 25 times and estimate the model using the WESBI method. For each of the different sampling proportions, we report the average posterior means and average posterior standard deviations of the parameter estimates across the 25 estimations.

We consider a total of 56 different sets of parameter values. To investigate the performance of the WESBI method on different network structures, we conduct experiments in two distinct categories of networks: networks with long tails and networks without long tails, as determined by the in-degree distribution. Because the long tail is a characteristic found in most online social networks, we use the first 32 experiments to show the performance of the WESBI method under various parameter combinations that lead to networks with long tails. For the following 24 experiments, we focus on the performance of the WESBI method for networks without long tails. The skewness of the in-degree distribution in our Epinions.com dataset lies within the range of skewness levels of the simulated networks we consider. This suggests that the WESBI method is appropriate to use for our research context.

Network Generation Process

We simulate networks by using a variation of the classic Barabasi and Albert (1999) model. There are initially m_0 isolated nodes in the network at time $t = 0$, and m nodes are added into the network in each time period for T time periods. Subsequently, we allow the network to evolve further by allowing the tie-formation process to continue for *K* additional time periods.

The expressions below specify the proportional hazard process governing the formation of a directed link from node i to node j:

$$\lambda_{ij} = \lambda_0 \exp(\beta_{1,i} z_{1,j} + \beta_{2,i} z_{2,j}),\ \lambda_0 > 0, \tag{1}$$

$$\boldsymbol{\beta}_i = \begin{bmatrix} \beta_{1,i} \\ \beta_{2,i} \end{bmatrix} = \boldsymbol{\delta} + \boldsymbol{\varepsilon_i} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \boldsymbol{\varepsilon_i},\ \boldsymbol{\varepsilon_i} \sim \text{MVN}(0, \boldsymbol{\Sigma}_\beta).$$

In the above, λ_0 is the baseline hazard rate which describes the inherent propensity of individual i forming a link with j without considering other factors and is independent of time. $z_{1,j}$ and $z_{2,j}$ are two different time-constant characteristics for individual j. Individual specific coefficients $\boldsymbol{\beta}_i$ capture how covariates have different impacts on individual tie-formation decisions across people. The quantity $\exp(\beta_{1,i} z_{1,j} + \beta_{2,i} z_{2,j})$ increases or decreases the baseline hazard rate of tie formation between i and j.

Simulation Design for Long-Tailed Networks

It has been observed that many complex networks, especially online social networks, have long-tailed degree distributions (Barabasi and Albert 1999; Mislove et al. 2007). As a result, it is especially important to study the performance of the WESBI model for networks with long tails.

To generate the networks, we set $m_0 = 1, m = 1$. To compare how our model adapts to networks of different sizes, we set $T \in \{2000, 5000\}$, resulting in networks of size 2001 or 5001, and set *K*=200. We set λ_0 to be a very small number so that the rate at which ties are formed is slow and the simulated network are relatively sparse. Specifically, we set $\lambda_0 \in \{e^{-50}, e^{-55}\}$ (i.e., $\log \lambda_0 \in \{-50, -55\}$). For the parameters, $\beta_{1,i}$ and $\beta_{2,i}$, we set $\delta_1 \in \{-2, -3\}, \delta_2 \in \{2,3\}$. The variance-covariance matrix of the coefficients are set to be the same across all simulated networks: $\boldsymbol{\Sigma_\beta} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \times 0.5$.

We employ two scenarios for the distributions of individual characteristics. In the first scenario, individual characteristics, $z_{1,j}$ and $z_{2,j}$ are drawn from two independent distributions: $z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$. In the second scenario, they are drawn from two independent distributions: $z_{1,j} \sim \text{N}(0, \sigma_z^2)$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$. Note that, in the first scenario the distribution for one covariate is skewed and the distribution for the other is symmetric, while in the second scenario both distributions are symmetric. While the exponential and the normal distributions themselves are not long-tail distributions, the hazard model we employ here allows us to construct networks in which the distributions of node degree are long-tailed. Intuitively, this is because the hazard rate of extending links is proportional to the exponent of the covariate values. Thus, the impact on tie formation of covariates with large values will be greatly magnified, implying that there will be individuals who have a large number of incoming links and will therefore contribute to a long tail. To support this argument, in the following sections, we report the mean and the max of the degrees of the various networks we generate, and compare these statistics across networks with and without long-tailed distributions. The parameter σ_z governs the variance of each distribution, and its value is set so that the generated networks are sparse.

We can vary the values of T, λ_0, δ_1, δ_2, and the distributions of the two covariates to generate networks with different characteristics. In this first simulation, we have $2^5 = 32$ different parameter combinations, i.e., we generate 32 different types of networks. As we can see from Table A2, the simulated networks cover a large variety of network structures. The size of the network is either 2001 or 5001, the maximum in-degree ranges from 441 to 1296, and the network density ranges from 0.007% to 0.078%. The low densities are representative of common online social networks such as the ones in our Epinions study. By setting the two factors $\delta_1$and δ_2 to have opposite mean impact ($\delta_1 \delta_2 < 0$), we can test the robustness of

WESBI on estimating parameters with different signs. Furthermore, we assume that different factors can have different average impact on the formation of ties ($|\delta_1| \neq |\delta_2|$). Thus we can also investigate how well WESBI estimates model parameters when one factor dominates the other by varying the values of δ_1 and δ_2. While we fix the variance-covariance matrix of individual heterogeneous parameters, by changing the values of δ_1 and δ_2, we can also illustrate how the relative values in the variance-covariance matrix, compared with the mean values of the parameters, will influence the estimation results.

The conditional-log-likelihood function for the data in our simulations simplifies to the following (the notation is described in the paper):

$$\log L = w_1\left(\sum_{(\mathbb{I}_{ij}=1)}\left\{\log\left[1-\exp\left\{-\lambda_0\exp\left(\beta_{1,i}z_{1,j}+\beta_{2,i}z_{2,j}\right)\right\}\right]-\lambda_0(k_{ij}-1)\cdot\exp\left(\beta_{1,i}z_{1,j}+\beta_{2,i}z_{2,j}\right)\right\}\right)$$

$$+w_0\left(\sum_{(\mathbb{I}_{ij}=0)}\left\{-\lambda_0(k_{ij}-1)\cdot\exp\left(\beta_{1,i}z_{1,j}+\beta_{2,i}z_{2,j}\right)\right\}\right)$$

Here, $w_0 = \frac{1-Q_1}{1-H_1}$ and $w_1 = \frac{Q_1}{H_1}$, where Q_1 is the fraction of the ties formed in the whole population, and H_1 is the fraction of the ties formed in the sampled dataset. For example, in a directed network with 2000 individuals, there are in total 3,998,000 possible pairs that can form a tie. If 2,000 ties are formed, $Q_1 = \frac{2000}{3998000} = 0.0005$. All observations where ties are formed are included in the sampled dataset, thus if we sample 100,000 pairs out of the 3,998,000 possible pairs, $H_1 = \frac{2000}{100,000} = 0.02$. This gives $w_0 = 1.0199$, and $w_1 = 0.025$. By varying the fraction of dyads with ties formed in the sampled dataset we can explore how the effectiveness of the WESBI method depends on the number of sampled observations. We pick three possible values of the sampling proportion: 5%, 10%, 15%, which means that we sample, respectively, 5%, 10% and 15% of the total number of dyad pairs that do not form ties. Note that we always sample all the ties that are formed. For each sampling proportion of each network, we repeatedly sample the network 25 times, each time to obtain the target sampling proportion. We then estimate the model 25 times on the 25 samples using the WESBI method. We report the average posterior means and the average posterior standard deviations of the parameter value estimates recovered from the 25 runs, and compare these results with the true parameter values that we used to generate the networks.

In summary, we are estimating the model on $32 \times 3 = 96$ different datasets, each 25 times. These 96 datasets show that the WESBI method recovers parameter values accurately and, therefore, works very well in a wide range of conditions.

Table A1: Parameter Values Used to Simulate Long-Tailed Networks

	T	$\log(\lambda_0)$	δ_1	δ_2	Distribution of Characteristics
Network 1	2000	-50	-2	2	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 2	2000	-50	-2	3	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 3	2000	-50	-3	2	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 4	2000	-50	-3	3	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 5	2000	-55	-2	2	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 6	2000	-55	-2	3	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 7	2000	-55	-3	2	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 8	2000	-55	-3	3	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 9	5000	-50	-2	2	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 10	5000	-50	-2	3	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 11	5000	-50	-3	2	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 12	5000	-50	-3	3	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 13	5000	-55	-2	2	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 14	5000	-55	-2	3	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 15	5000	-55	-3	2	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 16	5000	-55	-3	3	$z_{1,j} \sim \text{Exponential}(1) * \sigma_z$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 17	2000	-50	-2	2	$z_{1,j} \sim \text{N}(0, \sigma_z^2)$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 18	2000	-50	-2	3	$z_{1,j} \sim \text{N}(0, \sigma_z^2)$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$
Network 19	2000	-50	-3	2	$z_{1,j} \sim \text{N}(0, \sigma_z^2)$, $z_{2,j} \sim \text{N}(0, \sigma_z^2)$

Network 20	2000	-50	-3	3	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 21	2000	-55	-2	2	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 22	2000	-55	-2	3	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 23	2000	-55	-3	2	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 24	2000	-55	-3	3	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 25	5000	-50	-2	2	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 26	5000	-50	-2	3	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 27	5000	-50	-3	2	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 28	5000	-50	-3	3	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 29	5000	-55	-2	2	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 30	5000	-55	-2	3	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 31	5000	-55	-3	2	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$
Network 32	5000	-55	-3	3	$z_{1,j} \sim \mathrm{N}(0, \sigma_z^2)$, $z_{2,j} \sim \mathrm{N}(0, \sigma_z^2)$

Table A2: Statistics for the Long-Tailed Networks Simulated

	Number of ties formed in the simulated network	Mean of In-Degree	Max of In-Degree	Max/Mean Ratio of In-Degree	Network density
Network 1	2670	1.335	604	452.434	0.067%
Network 2	3079	1.540	682	443.001	0.077%
Network 3	2436	1.218	550	451.560	0.061%
Network 4	2472	1.236	619	500.809	0.062%
Network 5	1805	0.903	575	637.119	0.045%
Network 6	2611	1.306	535	409.805	0.065%
Network 7	1603	0.802	527	657.517	0.040%
Network 8	2840	1.420	729	513.380	0.071%
Network 9	2029	0.406	531	1308.526	0.008%
Network 10	2597	0.519	679	1307.278	0.010%
Network 11	2073	0.415	632	1524.361	0.008%
Network 12	2668	0.534	665	1246.252	0.011%
Network 13	1801	0.360	592	1643.531	0.007%
Network 14	3386	0.677	934	1379.209	0.014%
Network 15	2579	0.516	595	1153.548	0.010%
Network 16	2646	0.529	636	1201.814	0.011%
Network 17	3126	1.563	594	380.038	0.078%
Network 18	2693	1.347	441	327.516	0.067%
Network 19	2392	1.196	709	592.809	0.060%
Network 20	2767	1.384	960	693.892	0.069%
Network 21	2728	1.364	608	445.748	0.068%
Network 22	2448	1.224	659	538.399	0.061%
Network 23	1783	0.892	464	520.471	0.045%
Network 24	2393	1.197	563	470.539	0.060%
Network 25	2768	0.554	836	1510.116	0.011%
Network 26	3191	0.638	825	1292.698	0.013%
Network 27	2215	0.443	888	2004.515	0.009%
Network 28	2527	0.505	771	1525.524	0.010%
Network 29	2126	0.425	619	1456.471	0.009%

Network 30	3895	0.779	1296	1663.671	0.016%
Network 31	2621	0.524	881	1680.656	0.010%
Network 32	3887	0.777	1118	1438.127	0.016%

As we see from Table A2 above, while the mean in-degree for each of the 32 networks is less than two, the maximum in-degree for each network is two or three orders of magnitude larger. This is evident from the ratio between the max and the mean in-degree. For comparison, we simulate 500 scale-free networks with long-tailed in-degree distributions following the procedure in Barabasi and Albert (1999).[1] The mean of in-degrees across the 500 networks is 1.00, and the node with maximum in-degree across the 500 networks has an in-degree of 325. By comparing statistics of the scale-free networks with those of our 32 simulated networks, it is clear that the long tail property is more salient in the 32 networks we simulated.

The 32 tables that follow show the estimation results from our simulation study. Each table reports parameter estimates for one set of parameter values. In each table, the first column reports the true parameter values used to generate the 25 sample networks. The second column reports the parameter estimates when 5% of the dyads that did not form a tie were sampled and used for parameter estimation; for each parameter, we report the average posterior mean and, in parentheses, the average standard deviation across the 25 instances. We put a check mark adjacent to these numbers if the true value of the parameter falls within the 95% credible interval that is constructed by using the average posterior mean and the average posterior standard deviation. Similarly, we report the parameter estimates when 10% and 15% of the dyads that did not form a tie were sampled for parameter estimation in the fourth and sixth columns, respectively.

The results show that sampling 10% or 15% of the dyads that do not form a tie gives very accurate estimation results with the true parameter value *always* falling in the credible interval. Even when we sample only 5% of the total dyads that do not form a tie, the true network parameter value falls in the corresponding

[1] We start with one node in the network at time $t = 0$. Then, for T=5000 time periods, at each time period, we add one node with one tie that links the new node to one node already present in the network. The probability that a new node will link to node i depends on the degree of node i: $\Pr(\text{link to node } i) = \text{Degree}_i / \sum_j \text{Degree}_j$. Note that the degree distribution of a scale-free network follows a power law, which implies that it has a long tail.

95% credible interval approximately 75% of the time. These results show that we can estimate the parameters with high accuracy by sampling a relatively small fraction of the total network and using the WESBI method for estimation.

Network 1:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0420 (0.0248) ✓	-50.0158 (0.0226) ✓	-50.0067 (0.0208) ✓
δ_1	-2	-2.0428 (0.0191)	-2.0305 (0.0171) ✓	-2.0132 (0.0169) ✓
δ_2	2	2.0329 (0.0181) ✓	2.0244 (0.0174) ✓	2.0076 (0.0169) ✓
$\Sigma_{\beta,11}$	0.5	0.5268 (0.0214) ✓	0.5153 (0.0195) ✓	0.5108 (0.0185) ✓
$\Sigma_{\beta,12}$	0.25	0.2564 (0.0164) ✓	0.2532 (0.0161) ✓	0.2526 (0.0159) ✓
$\Sigma_{\beta,22}$	0.5	0.5257 (0.0192) ✓	0.5138 (0.0162) ✓	0.5117 (0.0157) ✓

Network 2

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0391 (0.0224) ✓	-50.0239 (0.0209) ✓	-49.9983 (0.0195) ✓
δ_1	-2	-1.9836 (0.0182) ✓	-1.9850 (0.0180) ✓	-1.9885 (0.0176) ✓
δ_2	3	3.0183 (0.0174) ✓	2. 9925 (0.0171) ✓	3.0038 (0.0167) ✓
$\Sigma_{\beta,11}$	0.5	0.5357 (0.0194) ✓	0.5202 (0.0189) ✓	0.5147 (0.0184) ✓
$\Sigma_{\beta,12}$	0.25	0.2534 (0.0150) ✓	0.2510 (0.0138) ✓	0.2508 (0.0136) ✓
$\Sigma_{\beta,22}$	0.5	0.5058 (0.0176) ✓	0.5030 (0.0173) ✓	0.5016 (0.0171) ✓

Network 3:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9681 (0.0226) ✓	-49.9746 (0.0217) ✓	-49.9804 (0.0203) ✓
δ_1	-3	-3.0312 (0.0184) ✓	-3.0268 (0.0183) ✓	-3.0192 (0.0180) ✓
δ_2	2	1.9602 (0.0186)	1.9738 (0.0184) ✓	1.9832 (0.0179) ✓
$\Sigma_{\beta,11}$	0.5	0.5279 (0.0187) ✓	0.5145 (0.0185) ✓	0.5073 (0.0183) ✓
$\Sigma_{\beta,12}$	0.25	0.2552 (0.0163) ✓	0.2549 (0.0158) ✓	0.2545 (0.0149) ✓
$\Sigma_{\beta,22}$	0.5	0.5089 (0.0178) ✓	0.5069 (0.0171) ✓	0.5056 (0.0164) ✓

Network 4:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0507 (0.0194)	-50.0305 (0.0190) ✓	-50.0177 (0.0182) ✓
δ_1	-3	-3.0479 (0.0185)	-3.0276 (0.0177) ✓	-3.0176 (0.0173) ✓
δ_2	3	2.9668 (0.0183) ✓	2.9727 (0.0174) ✓	2.9819 (0.0170) ✓
$\Sigma_{\beta,11}$	0.5	0.5304 (0.0196) ✓	0.5231 (0.0191) ✓	0.5165 (0.0187) ✓
$\Sigma_{\beta,12}$	0.25	0.2681 (0.0172) ✓	0.2658 (0.0168) ✓	0.2598 (0.0159) ✓
$\Sigma_{\beta,22}$	0.5	0.4813 (0.0176) ✓	0.4876 (0.0173) ✓	0.4915 (0.0167) ✓

Network 5:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-55	-54.8736 (0.0233) ✓	-54.8966 (0.0219) ✓	-54.9361 (0.0192) ✓
δ_1	-2	-2.0140 (0.0194) ✓	-2.0115 (0.0184) ✓	-2.0080 (0.0179) ✓
δ_2	2	2.0395 (0.0182)	2.0288 (0.0180) ✓	2.0152 (0.0172) ✓
$\Sigma_{\beta,11}$	0.5	0.5376 (0.0187)	0.5216 (0.0183) ✓	0.5148 (0.0182) ✓
$\Sigma_{\beta,12}$	0.25	0.2612 (0.0171) ✓	0.2601 (0.0167) ✓	0.2585 (0.0156) ✓
$\Sigma_{\beta,22}$	0.5	0.5126 (0.0184) ✓	0.5095 (0.0172) ✓	0.5086 (0.0164) ✓

Network 6:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-55	-55.0291 (0.0204) ✓	-54.9833 (0.0199) ✓	-55.0084 (0.0199) ✓
δ_1	-2	-1.9850 (0.0185) ✓	-1.9877 (0.0181) ✓	-1.9928 (0.0165) ✓
δ_2	3	3.0261 (0.0177) ✓	3.0255 (0.0176) ✓	3.0173 (0.0169) ✓
$\Sigma_{\beta,11}$	0.5	0. 4807 (0.0192) ✓	0. 4815 (0.0180) ✓	0. 4862 (0.0174) ✓
$\Sigma_{\beta,12}$	0.25	0.2426 (0.0151) ✓	0.2442 (0.0146) ✓	0.2447 (0.0140) ✓
$\Sigma_{\beta,22}$	0.5	0.5162 (0.0176) ✓	0.5115 (0.0168) ✓	0.5107 (0.0165) ✓

Network 7:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-55	-55.0329 (0.0216) ✓	-55.0279 (0.0203) ✓	-55.0138 (0.0183) ✓
δ_1	-3	-2.9613 (0.0184)	-2.9796 (0.0177) ✓	-2.9853 (0.0173) ✓
δ_2	2	1.9710 (0.0184) ✓	1.9763 (0.0180) ✓	1.9810 (0.0176) ✓
$\Sigma_{\beta,11}$	0.5	0.4857 (0.0183) ✓	0.4872 (0.0180) ✓	0.4937 (0.0173) ✓
$\Sigma_{\beta,12}$	0.25	0.2566 (0.0169) ✓	0.2460 (0.0164) ✓	0.2532 (0.0156) ✓
$\Sigma_{\beta,22}$	0.5	0.5291 (0.0176) ✓	0.5230 (0.0170) ✓	0.5184 (0.0168) ✓

Network 8:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-55	-55.0385 (0.0286) ✓	-55.0295 (0.0264) ✓	-55.0207 (0.0253) ✓
δ_1	-3	-3.0310 (0.0187) ✓	-3.0193 (0.0182) ✓	-3.0102 (0.0175) ✓
δ_2	3	3.0312 (0.0175) ✓	3.0236 (0.0172) ✓	3.0120 (0.0168) ✓
$\Sigma_{\beta,11}$	0.5	0.4877 (0.0185) ✓	0.4893 (0.0174) ✓	0.4918 (0.0169) ✓
$\Sigma_{\beta,12}$	0.25	0.2578 (0.0182) ✓	0.2563 (0.0174) ✓	0.2558 (0.0169) ✓
$\Sigma_{\beta,22}$	0.5	0.4863 (0.0167) ✓	0.5036 (0.0163) ✓	0.4973 (0.0157) ✓

Network 9:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9711 (0.0184) ✓	-49.9778 (0.0174) ✓	-49.9820 (0.0168) ✓
δ_1	-2	-2.0245 (0.0117)	-2.0191 (0.0112) ✓	-2.0159 (0.0109) ✓
δ_2	2	2.0283 (0.0122)	2.0183 (0.0114) ✓	2.0146 (0.0112) ✓
$\Sigma_{\beta,11}$	0.5	0.5148 (0.0116) ✓	0.5103 (0.0115) ✓	0.5086 (0.0109) ✓
$\Sigma_{\beta,12}$	0.25	0.2583 (0.0103) ✓	0.2567 (0.0097) ✓	0.2539 (0.0097) ✓
$\Sigma_{\beta,22}$	0.5	0.5134 (0.0115) ✓	0.5084 (0.0112) ✓	0.5079 (0.0109) ✓

Network 10:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0279 (0.0178) ✓	-50.0250 (0.0173) ✓	-50.0167 (0.0169) ✓
δ_1	-2	-2.0275 (0.0116)	-2.0193 (0.0115) ✓	-2.0174 (0.0114) ✓
δ_2	3	3.0178 (0.0115) ✓	3.0157 (0.0112) ✓	3.0122 (0.0111) ✓
$\Sigma_{\beta,11}$	0.5	0.5135 (0.0129) ✓	0.5078 (0.0124) ✓	0.5064 (0.0119) ✓
$\Sigma_{\beta,12}$	0.25	0.2611 (0.0086) ✓	0.2594 (0.0086) ✓	0.2583 (0.0082) ✓
$\Sigma_{\beta,22}$	0.5	0.5137 (0.0114) ✓	0.5124 (0.0106) ✓	0.5084 (0.0106) ✓

Network 11:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0395 (0.0185)	-50.0216 ✓ (0.0183)	-50.0141 ✓ (0.0178)
δ_1	-3	-3.0218 ✓ (0.0123)	-3.0135 ✓ (0.0119)	-3.0063 ✓ (0.0118)
δ_2	2	2.0256 (0.0119)	2.0158 ✓ (0.0115)	2.0126 ✓ (0.0109)
$\Sigma_{\beta,11}$	0.5	0.5122 ✓ (0.0116)	0.5109 ✓ (0.0114)	0.5089 ✓ (0.0114)
$\Sigma_{\beta,12}$	0.25	0.2561 ✓ (0.0106)	0.2558 ✓ (0.0104)	0.2556 ✓ (0.0098)
$\Sigma_{\beta,22}$	0.5	0.5160 ✓ (0.0112)	0.5138 ✓ (0.0110)	0.5112 ✓ (0.0110)

Network 12:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9653 ✓ (0.0190)	-49.9775 ✓ (0.0186)	-49.9831 ✓ (0.0183)
δ_1	-3	-3.0260 (0.0118)	-3.0150 ✓ (0.0115)	-3.0126 ✓ (0.0115)
δ_2	3	3.0140 ✓ (0.0117)	3.0102 ✓ (0.0115)	3.0088 ✓ (0.0114)
$\Sigma_{\beta,11}$	0.5	0.5069 ✓ (0.0114)	0.5061 ✓ (0.0113)	0.5057 ✓ (0.0110)
$\Sigma_{\beta,12}$	0.25	0.2549 ✓ (0.0097)	0.2535 ✓ (0.0097)	0.2496 ✓ (0.0097)
$\Sigma_{\beta,22}$	0.5	0.5197 ✓ (0.0113)	0.5123 ✓ (0.0112)	0.5086 ✓ (0.0110)

Network 13:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-54.9694 (0.0192)	✓	-54.9821 (0.0183)	✓	-54.9878 (0.0177)	✓
δ_1	-2	-2.0291 (0.0116)		-2.0184 (0.0115)	✓	-2.0123 (0.0113)	✓
δ_2	2	2.0215 (0.0121)	✓	2.0165 (0.0116)	✓	2.0108 (0.0114)	✓
$\Sigma_{\beta,11}$	0.5	0.5194 (0.0118)	✓	0.5089 (0.0117)	✓	0.5062 (0.0115)	✓
$\Sigma_{\beta,12}$	0.25	0.2529 (0.0096)	✓	0.2516 (0.0095)	✓	0.2497 (0.0095)	✓
$\Sigma_{\beta,22}$	0.5	0.5121 (0.0117)	✓	0.5093 (0.0116)	✓	0.5064 (0.0113)	✓

Network 14:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-55.0399 (0.0186)		-55.0218 (0.0181)	✓	-55.0169 (0.0176)	✓
δ_1	-2	-1.9731 (0.0120)		-1.9812 (0.0116)	✓	-1.9884 (0.0113)	✓
δ_2	3	3.0127 (0.0114)	✓	3.0073 (0.0114)	✓	3.0054 (0.0112)	✓
$\Sigma_{\beta,11}$	0.5	0. 5165 (0.0116)	✓	0. 5114 (0.0112)	✓	0. 5065 (0.0110)	✓
$\Sigma_{\beta,12}$	0.25	0.2417 (0.0103)	✓	0.2441 (0.0100)	✓	0.2449 (0.0097)	✓
$\Sigma_{\beta,22}$	0.5	0.5089 (0.0118)	✓	0.5079 (0.0114)	✓	0.5065 (0.0112)	✓

Network 15:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-55.0314 (0.0194)	✓	-55.0176 (0.0187)	✓	-55.0086 (0.0180)	✓
δ_1	-3	-3.0145 (0.0120)	✓	-3.0116 (0.0117)	✓	-3.0088 (0.0116)	✓
δ_2	2	2.0176 (0.0113)	✓	2.0128 (0.0113)	✓	2.0076 (0.0111)	✓
$\Sigma_{\beta,11}$	0.5	0.5092 (0.0118)	✓	0.5089 (0.0116)	✓	0.5033 (0.0114)	✓
$\Sigma_{\beta,12}$	0.25	0.2590 (0.0103)	✓	0.2584 (0.0100)	✓	0.2563 (0.0097)	✓
$\Sigma_{\beta,22}$	0.5	0.5128 (0.0110)	✓	0.5102 (0.0107)	✓	0.5072 (0.0104)	✓

Network 16:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-54.9583 (0.0193)		-54.9812 (0.0184)	✓	-54.9843 (0.0181)	✓
δ_1	-3	-3.0314 (0.0119)		-3.0184 (0.0117)	✓	-3.0138 (0.0115)	✓
δ_2	3	2.9743 (0.0116)		2.9824 (0.0114)	✓	2.9875 (0.0114)	✓
$\Sigma_{\beta,11}$	0.5	0.5180 (0.0114)	✓	0.5124 (0.0113)	✓	0.5089 (0.0110)	✓
$\Sigma_{\beta,12}$	0.25	0.2579 (0.0097)	✓	0.2573 (0.0096)	✓	0.2553 (0.0093)	✓
$\Sigma_{\beta,22}$	0.5	0.5156 (0.0117)	✓	0.5083 (0.0114)	✓	0.5068 (0.0113)	✓

Network 17:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0468 (0.0259)	✓	-50.0285 (0.0232)	✓	-50.0208 (0.0207)	✓
δ_1	-2	-2.0362 (0.0195)	✓	-2.0278 (0.0186)	✓	-2.0132 (0.0182)	✓
δ_2	2	2.0321 (0.0190)	✓	2.0208 (0.0185)	✓	2.0132 (0.0181)	✓
$\Sigma_{\beta,11}$	0.5	0.5348 (0.0207)	✓	0.5235 (0.0191)	✓	0.5176 (0.0183)	✓
$\Sigma_{\beta,12}$	0.25	0.2423 (0.0159)	✓	0.2452 (0.0156)	✓	0.2466 (0.0151)	✓
$\Sigma_{\beta,22}$	0.5	0.5195 (0.0205)	✓	0.5164 (0.0196)	✓	0.5103 (0.0173)	✓

Network 18:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0573 (0.0236)		-50.0287 (0.0221)	✓	-50.0185 (0.0209)	✓
δ_1	-2	-2.0277 (0.0192)	✓	-2.0098 (0.0187)	✓	-2.0034 (0.0183)	✓
δ_2	3	3.0397 (0.0182)		3.0169 (0.0179)	✓	3.0144 (0.0171)	✓
$\Sigma_{\beta,11}$	0.5	0.5295 (0.0185)	✓	0.5208 (0.0181)	✓	0.5124 (0.0175)	✓
$\Sigma_{\beta,12}$	0.25	0.2622 (0.0159)	✓	0.2576 (0.0147)	✓	0.2541 (0.0140)	✓
$\Sigma_{\beta,22}$	0.5	0.5167 (0.0175)	✓	0.5134 (0.0172)	✓	0.5065 (0.0171)	✓

Network 19:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0271 (0.0218)	✓	-50.0176 (0.0204)	✓	-50.0117 (0.0196)	✓
δ_1	-3	-3.0456 (0.0192)		-3.0187 (0.0187)	✓	-3.0145 (0.0183)	✓
δ_2	2	2.0207 (0.0198)	✓	2.0164 (0.0193)	✓	2.0117 (0.0185)	✓
$\Sigma_{\beta,11}$	0.5	0.4734 (0.0186)	✓	0.4895 (0.0181)	✓	0.4944 (0.0174)	✓
$\Sigma_{\beta,12}$	0.25	0.2611 (0.0156)	✓	0.2570 (0.0153)	✓	0.2561 (0.0150)	✓
$\Sigma_{\beta,22}$	0.5	0.4761 (0.0182)	✓	0.4820 (0.0181)	✓	0.4788 (0.0173)	✓

Network 20:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0502 (0.0213)		-50.0278 (0.0197)	✓	-49.9950 (0.0191)	✓
δ_1	-3	-3.0529 (0.0172)		-3.0306 (0.0168)	✓	-3.0208 (0.0164)	✓
δ_2	3	3.0502 (0.0182)		3.0314 (0.0180)	✓	3.0239 (0.0167)	✓
$\Sigma_{\beta,11}$	0.5	0.4965 (0.0177)	✓	0.4972 (0.0175)	✓	0.4998 (0.0165)	✓
$\Sigma_{\beta,12}$	0.25	0.2682 (0.0171)	✓	0.2637 (0.0168)	✓	0.2572 (0.0157)	✓
$\Sigma_{\beta,22}$	0.5	0.5174 (0.0188)	✓	0.5155 (0.0184)	✓	0.5084 (0.0181)	✓

Network 21:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-55	-54.4894 (0.0227)	-54.8065 ✓ (0.0221)	-54.8543 ✓ (0.0202)
δ_1	-2	-2.0308 ✓ (0.0191)	-2.0207 ✓ (0.0187)	-2.0143 ✓ (0.0182)
δ_2	2	2.0298 ✓ (0.0187)	2.0230 ✓ (0.0179)	2.0186 ✓ (0.0172)
$\Sigma_{\beta,11}$	0.5	0.5138 ✓ (0.0191)	0.5117 ✓ (0.0187)	0.5063 ✓ (0.0181)
$\Sigma_{\beta,12}$	0.25	0.2543 ✓ (0.0168)	0.2524 ✓ (0.0162)	0.2485 ✓ (0.0159)
$\Sigma_{\beta,22}$	0.5	0.5163 ✓ (0.0185)	0.5112 ✓ (0.0176)	0.5076 ✓ (0.0169)

Network 22:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-55	-55.0581 (0.0214)	-55.0407 ✓ (0.0212)	-55.0157 ✓ (0.0198)
δ_1	-2	-2.0308 ✓ (0.0175)	-2.0249 ✓ (0.0173)	-3.0206 ✓ (0.0167)
δ_2	3	3.0390 (0.0175)	3.0303 ✓ (0.0172)	3.0262 ✓ (0.0165)
$\Sigma_{\beta,11}$	0.5	0.5170 ✓ (0.0179)	0.5136 ✓ (0.0172)	0.5117 ✓ (0.0167)
$\Sigma_{\beta,12}$	0.25	0.2708 ✓ (0.0142)	0.2693 ✓ (0.0132)	0.2668 ✓ (0.0130)
$\Sigma_{\beta,22}$	0.5	0.5216 ✓ (0.0181)	0.5203 ✓ (0.0174)	0.5173 ✓ (0.0167)

Network 23:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-54.9529 (0.0225)		-54.9727 (0.0208)	✓	-54.9858 (0.0196)	✓
δ_1	-3	-3.0094 (0.0184)	✓	-3.0049 (0.0173)	✓	-2.9994 (0.0169)	✓
δ_2	2	2.0178 (0.0177)	✓	2.0088 (0.0176)	✓	2.0032 (0.0172)	✓
$\Sigma_{\beta,11}$	0.5	0.5189 (0.0187)	✓	0.5148 (0.0182)	✓	0.5128 (0.0177)	✓
$\Sigma_{\beta,12}$	0.25	0.2418 (0.0141)	✓	0.2429 (0.0139)	✓	0.2441 (0.0136)	✓
$\Sigma_{\beta,22}$	0.5	0.5072 (0.0183)	✓	0.5022 (0.0178)	✓	0.5011 (0.0171)	✓

Network 24:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-55.0519 (0.0243)		-55.0291 (0.0215)	✓	-55.0159 (0.0202)	✓
δ_1	-3	-3.0372 (0.0193)	✓	-3.0244 (0.0182)	✓	-3.0145 (0.0177)	✓
δ_2	3	3.0138 (0.0182)	✓	3.0105 (0.0175)	✓	3.0074 (0.0171)	✓
$\Sigma_{\beta,11}$	0.5	0.5125 (0.0183)	✓	0.5098 (0.0179)	✓	0.5053 (0.0174)	✓
$\Sigma_{\beta,12}$	0.25	0.2407 (0.0178)	✓	0.2446 (0.0176)	✓	0.2472 (0.0171)	✓
$\Sigma_{\beta,22}$	0.5	0.5426 (0.0175)		0.5214 (0.0171)	✓	0.5133 (0.0168)	✓

Network 25:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-49.9575 (0.0192)		-49.9754 (0.0186)	✓	-49.9840 (0.0174)	✓
δ_1	-2	-2.0281 (0.0120)		-2.0204 (0.0117)	✓	-2.0187 (0.0114)	✓
δ_2	2	2.0299 (0.0119)		2.0187 (0.0115)	✓	2.0140 (0.0111)	✓
$\Sigma_{\beta,11}$	0.5	0.5186 (0.0121)	✓	0.5154 (0.0117)	✓	0.5145 (0.0117)	✓
$\Sigma_{\beta,12}$	0.25	0.2619 (0.0105)	✓	0.2586 (0.0101)	✓	0.2572 (0.0097)	✓
$\Sigma_{\beta,22}$	0.5	0.5255 (0.0114)	✓	0.5184 (0.0111)	✓	0.5127 (0.0109)	✓

Network 26:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-49.9550 (0.0191)		-49.9703 (0.0163)	✓	-49.9982 (0.0144)	✓
δ_1	-2	-2.0157 (0.0115)	✓	-2.0133 (0.0113)	✓	-2.0114 (0.0103)	✓
δ_2	3	3.0288 (0.0118)		3.0201 (0.0112)	✓	3.0169 (0.0111)	✓
$\Sigma_{\beta,11}$	0.5	0.5159 (0.0121)	✓	0.5142 (0.0115)	✓	0.5126 (0.0113)	✓
$\Sigma_{\beta,12}$	0.25	0.2464 (0.0099)	✓	0.2477 (0.0091)	✓	0.2482 (0.0090)	✓
$\Sigma_{\beta,22}$	0.5	0.5025 (0.0119)	✓	0.4975 (0.0116)	✓	0.4987 (0.0109)	✓

Network 27:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9730 (0.0236) ✓	-49.9828 (0.0207) ✓	-49.9893 (0.0183) ✓
δ_1	-3	-3.0251 (0.0119)	-3.0207 (0.0111) ✓	-3.0196 (0.0109) ✓
δ_2	2	2.0078 (0.0115) ✓	2.0077 (0.0111) ✓	2.0002 (0.0106) ✓
$\Sigma_{\beta,11}$	0.5	0.4908 (0.0119) ✓	0.4942 (0.0117) ✓	0.4969 (0.0111) ✓
$\Sigma_{\beta,12}$	0.25	0.2425 (0.0099) ✓	0.2433 (0.0094) ✓	0.2442 (0.0088) ✓
$\Sigma_{\beta,22}$	0.5	0.4939 (0.0116) ✓	0.4974 (0.0113) ✓	0.4985 (0.0107) ✓

Network 28:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0160 (0.0194) ✓	-50.0126 (0.0190) ✓	-50.0087 (0.0184) ✓
δ_1	-3	-3.0235 (0.0117)	-3.0194 (0.0112) ✓	-3.0126 (0.0107) ✓
δ_2	3	3.0271 (0.0115)	3.0197 (0.0111) ✓	3.0137 (0.0108) ✓
$\Sigma_{\beta,11}$	0.5	0.5284 (0.0124)	0.5142 (0.0120) ✓	0.5095 (0.0115) ✓
$\Sigma_{\beta,12}$	0.25	0.2592 (0.0104) ✓	0.2558 (0.0101) ✓	0.2476 (0.0093) ✓
$\Sigma_{\beta,22}$	0.5	0.5201 (0.0115) ✓	0.5193 (0.0111) ✓	0.5135 (0.0109) ✓

Network 29:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-54.9721 (0.0200)	✓	-54.9794 (0.0192)	✓	-54.9853 (0.0185)	✓
δ_1	-2	-2.0294 (0.0121)		-2.0204 (0.0117)	✓	-2.0158 (0.0112)	✓
δ_2	2	2.0322 (0.0116)		2.0184 (0.0113)	✓	2.0116 (0.0111)	✓
$\Sigma_{\beta,11}$	0.5	0.5183 (0.0120)	✓	0.5120 (0.0115)	✓	0.5062 (0.0109)	✓
$\Sigma_{\beta,12}$	0.25	0.2423 (0.0093)	✓	0.2458 (0.0092)	✓	0.2477 (0.0092)	✓
$\Sigma_{\beta,22}$	0.5	0.5159 (0.0119)	✓	0.5137 (0.0117)	✓	0.5087 (0.0112)	✓

Network 30:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-55.0402 (0.0193)		-55.0311 (0.0191)	✓	-55.0248 (0.0186)	✓
δ_1	-2	-1.9718 (0.0118)		-1.9862 (0.0113)	✓	-1.9913 (0.0110)	✓
δ_2	3	2.9769 (0.0114)		2.9820 (0.0111)	✓	2.9897 (0.0110)	✓
$\Sigma_{\beta,11}$	0.5	0. 5072 (0.0120)	✓	0. 5026 (0.0113)	✓	0. 4992 (0.0107)	✓
$\Sigma_{\beta,12}$	0.25	0.2588 (0.0103)	✓	0.2523 (0.0101)	✓	0.2497 (0.0097)	✓
$\Sigma_{\beta,22}$	0.5	0.5102 (0.0114)	✓	0.5093 (0.0112)	✓	0.5064 (0.0111)	✓

Network 31:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-55.0165 (0.0202)	✓	-55.0133 (0.0190)	✓	-55.0083 (0.0180)	✓
δ_1	-3	-3.0221 (0.0121)	✓	-3.0168 (0.0115)	✓	-3.0107 (0.0113)	✓
δ_2	2	2.0269 (0.0123)		2.0205 (0.0116)	✓	2.0154 (0.0108)	✓
$\Sigma_{\beta,11}$	0.5	0.5161 (0.0123)	✓	0.5116 (0.0115)	✓	0.5074 (0.0114)	✓
$\Sigma_{\beta,12}$	0.25	0.2428 (0.0104)	✓	0.2469 (0.0101)	✓	0.2478 (0.0095)	✓
$\Sigma_{\beta,22}$	0.5	0.5080 (0.0106)	✓	0.5051 (0.0102)	✓	0.5021 (0.0098)	✓

Network 32:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-55	-55.0441 (0.0202)		-55.0259 (0.0196)	✓	-55.0187 (0.0182)	✓
δ_1	-3	-2.9730 (0.0124)		-2.9879 (0.0118)	✓	-2.9930 (0.0110)	✓
δ_2	3	2.9739 (0.0119)		2.9819 (0.0112)	✓	2.9883 (0.0111)	✓
$\Sigma_{\beta,11}$	0.5	0.5107 (0.0110)	✓	0.5077 (0.0107)	✓	0.5020 (0.0102)	✓
$\Sigma_{\beta,12}$	0.25	0.2425 (0.0093)	✓	0.2458 (0.0092)	✓	0.2484 (0.0092)	✓
$\Sigma_{\beta,22}$	0.5	0.5182 (0.0115)	✓	0.5082 (0.0113)	✓	0.5037 (0.0111)	✓

Simulation Design for Non-Long-Tailed Networks

All generated networks in the preceding simulation exercise have a long-tailed degree distribution. While this characteristic is present in most online social networks, it is nonetheless important to demonstrate the performance of WESBI on networks that are of "short" tail. We conduct this exercise now. Following the same simulation scheme described above for networks with long tails, we vary the values of T, δ_1, δ_2, and the distributions of the two covariates to generate networks with different characteristics. Notably, we choose distributions for the individual characteristics, $z_{1,j}$ and $z_{2,j}$, such that the generated networks do not have long-tailed degree distributions.

Specifically, we consider three scenarios. In the first scenario, $z_{1,j}$ and $z_{2,j}$ are drawn from the following independent distributions: $z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_z$, $z_{2,j} \sim \text{Uniform}(-1,1) * \sigma_z$. In the second scenario, $z_{1,j}$ and $z_{2,j}$ are drawn from the following independent distributions: $z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_z + \mu_{z1}$, $z_{2,j} \sim \text{Uniform}(-1,1) * \sigma_z + \mu_{z2}$, where $\mu_{z1} < 0$ and $\mu_{z2} > 0$. In the third scenario, $z_{1,j}$ and $z_{2,j}$ are drawn from the following independent distributions: $z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_{z_1}$, $z_{2,j} \sim \text{Log-normal}(0,1) * \sigma_{z_2}$. Note that in the first scenario the value of individual characteristics are bounded from above and the distribution is uniform, and thus we will not observe individuals with an exceptionally large number of incoming links. Correspondingly, networks generated according to the first scenario will not be long-tailed. In the second scenario, we assign a small value to σ_z and set $\mu_{z1} < 0$ and $\mu_{z2} > 0$, which simulates networks which are significantly less heterogeneous and less skewed, i.e., with shorter tails, compared with those in the first scenario. In the third scenario, we consider "hybrid" cases in which the distribution of one individual characteristic is uniform and bounded from above, while the distribution of the other individual characteristic is skewed and not bounded from above (by virtue of being log-normally distributed). In a later part of this section, we use some statistics to show this "short tail" property in networks generated in these three scenarios.

In the simulation in this section, we have $2^3 \times 3 = 24$ different parameter combinations, thus we generate 24 different types of networks. As we can see from Table A4, the simulated networks cover a large variety of network structures. The size of the network can be either 2001 or 5001, the maximum in-degree ranges from 9 to 347, and the network density ranges from 0.007% to 0.090%. Table A4 shows some statistics of the networks that are generated in these scenarios. As we can see, the maximum in-degrees of the 24 networks generated in these scenarios are significantly smaller than the maximum in-degrees of the 32 long-tail networks in Table A2. This suggests that the networks studied in this section have relatively "short" tails.

For further comparison, we plot the in-degree distributions of representative short- and long-tailed networks in Figure A1. We use data from Network 45 as an example of a short-tail network, and Network 25 as an example of a long-tail network. The *x*-axis shows the in-degree (exact in-degree when this value is ≤3, and a range for larger in-degree, with the range progressively increasing), and the *y*-axis shows the frequency on a log scale. As we can see from Figure A1, the tail of the in-degree distribution for the short-tail network disappears even for small in-degree (the maximum in-degree is 9 in this case), while the tail of the in-degree distribution for the long-tail network extends to much larger numbers (the maximum in-degree is 836 in this case). By combining generated networks from all scenarios with both long tail and short tail, our simulation study covers a wide range of network structures in terms of their thickness of the tails of the in-degree distributions.

Table A3: Parameter Values Used to Simulate Non-Long-Tailed Networks

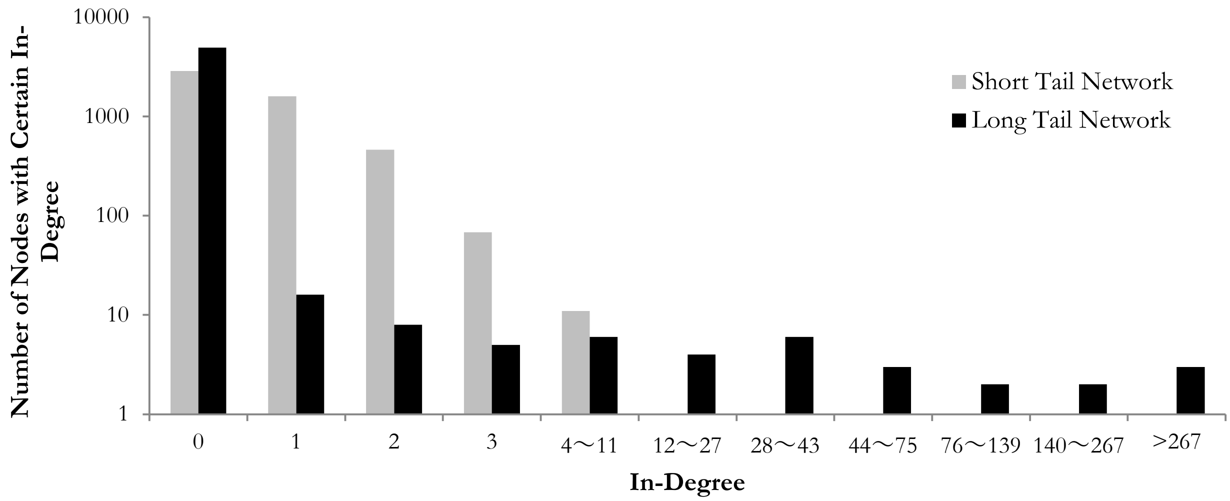
	T	$\log(\lambda_0)$	δ_1	δ_2	Distribution of Characteristics
Network 33	2000	-50	-2	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_z$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z$
Network 34	2000	-50	-2	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_z$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z$
Network 35	2000	-50	-3	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_z$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z$
Network 36	2000	-50	-3	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_z$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z$
Network 37	5000	-50	-2	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_z$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z$
Network 38	5000	-50	-2	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_z$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z$
Network 39	5000	-50	-3	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_z$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z$
Network 40	5000	-50	-3	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_z$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z$
Network 41	2000	-50	-2	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z1}$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z2}$
Network 42	2000	-50	-2	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z1}$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z2}$
Network 43	2000	-50	-3	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z1}$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z2}$
Network 44	2000	-50	-3	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z1}$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z2}$
Network 45	5000	-50	-2	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z1}$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z2}$
Network 46	5000	-50	-2	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z1}$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z2}$
Network 47	5000	-50	-3	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z1}$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z2}$
Network 48	5000	-50	-3	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z1}$, $z_{2,j}$~Uniform$(-1,1)*\sigma_z+\mu_{z2}$
Network 49	2000	-50	-2	2	$z_{1,j}$~Uniform$(-1,1)*\sigma_{z_1}$, $z_{2,j}$~Log-normal$(0,1)*\sigma_{z_2}$
Network 50	2000	-50	-2	3	$z_{1,j}$~Uniform$(-1,1)*\sigma_{z_1}$, $z_{2,j}$~Log-normal$(0,1)*\sigma_{z_2}$

Network 51	2000	-50	-3	2	$z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_{z_1}$, $z_{2,j} \sim \text{Log-normal}(0,1) * \sigma_{z_2}$
Network 52	2000	-50	-3	3	$z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_{z_1}$, $z_{2,j} \sim \text{Log-normal}(0,1) * \sigma_{z_2}$
Network 53	5000	-50	-2	2	$z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_{z_1}$, $z_{2,j} \sim \text{Log-normal}(0,1) * \sigma_{z_2}$
Network 54	5000	-50	-2	3	$z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_{z_1}$, $z_{2,j} \sim \text{Log-normal}(0,1) * \sigma_{z_2}$
Network 55	5000	-50	-3	2	$z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_{z_1}$, $z_{2,j} \sim \text{Log-normal}(0,1) * \sigma_{z_2}$
Network 56	5000	-50	-3	3	$z_{1,j} \sim \text{Uniform}(-1,1) * \sigma_{z_1}$, $z_{2,j} \sim \text{Log-normal}(0,1) * \sigma_{z_2}$

Table A4: Statistics for the Non-Long-Tailed Networks Simulated

	Number of ties formed in the simulated network	Mean of In-Degree	Max of In-Degree	Max/Mean Ratio of In-Degree	Network density
Network 33	2087	1.0430	63	60.404	0.052%
Network 34	1774	0.8866	54	60.910	0.044%
Network 35	3259	1.6287	102	62.627	0.081%
Network 36	2839	1.4188	86	60.615	0.071%
Network 37	2254	0.4507	37	82.093	0.009%
Network 38	2021	0.4041	39	96.506	0.008%
Network 39	2490	0.4979	32	64.270	0.010%
Network 40	2530	0.5059	35	69.184	0.010%
Network 41	2176	1.087	13	11.955	0.054%
Network 42	2523	1.261	11	8.724	0.063%
Network 43	2861	1.430	15	10.491	0.071%
Network 44	3604	1.801	16	8.883	0.090%
Network 45	2786	0.557	9	16.155	0.011%
Network 46	3173	0.634	15	23.642	0.013%
Network 47	2991	0.598	18	30.096	0.012%
Network 48	2604	0.521	11	21.126	0.010%
Network 49	2932	1.465	128	87.356	0.073%
Network 50	2269	1.134	70	61.732	0.057%
Network 51	2036	1.017	58	57.030	0.051%
Network 52	3241	1.620	163	100.637	0.081%
Network 53	2491	0.498	245	491.869	0.010%
Network 54	2604	0.521	175	336.089	0.010%
Network 55	2862	0.572	221	386.171	0.011%
Network 56	3040	0.608	242	398.106	0.012%

Figure A1: In-Degree Distributions for Representative Short- and Long-Tail Networks



For each target value of the sampling proportion (5%, 10%, 15% of the total dyads that do not form a tie) of the 24 networks, we repeatedly sample the network 25 times, and estimate the model on the 25 samples using the WESBI method. We report the average posterior means and the average posterior standard deviations of the parameter values recovered from the 25 runs, and check whether the true network generating parameters fall within the 95% credible intervals that are constructed using these values.

The 24 tables that follow show the estimation results, presented as before, from our simulation study. A check mark indicates that the true value of the parameter falls within the 95% credible interval that is constructed by using the average posterior mean and the average posterior standard deviation using estimates from the 25 runs. As before, the results show that sampling 10% or 15% of the dyads that do not form a tie gives very accurate estimation results with the true parameter value *always* falling in the credible interval. This shows that we can estimate the parameters with high accuracy by sampling a relatively small fraction of the total network and using the WESBI method for estimation.

Network 33:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0463 (0.0251) ✓	-50.0293 (0.0221) ✓	-50.0163 (0.0196) ✓
δ_1	-2	-1.9669 (0.0173) ✓	-1.9765 (0.0161) ✓	-1.9811 (0.0154) ✓
δ_2	2	1.9517 (0.0176)	1.9718 (0.0171) ✓	1.9853 (0.0160) ✓
$\Sigma_{\beta,11}$	0.5	0.5194 (0.0213) ✓	0.5132 (0.0201) ✓	0.5081 (0.0185) ✓
$\Sigma_{\beta,12}$	0.25	0.2581 (0.0161) ✓	0.2568 (0.0157) ✓	0.2540 (0.0146) ✓
$\Sigma_{\beta,22}$	0.5	0.5126 (0.0194) ✓	0.5111 (0.0185) ✓	0.5093 (0.0180) ✓

Network 34:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9849 (0.0216) ✓	-49.9903 (0.0204) ✓	-49.9941 (0.0196) ✓
δ_1	-2	-2.0133 (0.0183) ✓	-2.0102 (0.0177) ✓	-2.0075 (0.0164) ✓
δ_2	3	3.0477 (0.0193)	3.0179 (0.0181) ✓	3.0081 (0.0172) ✓
$\Sigma_{\beta,11}$	0.5	0.5179 (0.0187) ✓	0.5139 (0.0174) ✓	0.5091 (0.0165) ✓
$\Sigma_{\beta,12}$	0.25	0.2457 (0.0156) ✓	0.2471 (0.0143) ✓	0.2489 (0.0140) ✓
$\Sigma_{\beta,22}$	0.5	0.5237 (0.0176) ✓	0.5153 (0.0163) ✓	0.5101 (0.0160) ✓

Network 35:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-49.9739 (0.0226)	✓	-49.9801 (0.0215)	✓	-49.9864 (0.0185)	✓
δ_1	-3	-3.0089 (0.0165)	✓	-3.0050 (0.0182)	✓	-3.0020 (0.0169)	✓
δ_2	2	2.0134 (0.0181)	✓	2.0100 (0.0178)	✓	2.0071 (0.0171)	✓
$\Sigma_{\beta,11}$	0.5	0.4863 (0.0179)	✓	0.4870 (0.0178)	✓	0.4880 (0.0172)	✓
$\Sigma_{\beta,12}$	0.25	0.2321 (0.0143)	✓	0.2394 (0.0143)	✓	0.2429 (0.0140)	✓
$\Sigma_{\beta,22}$	0.5	0.4871 (0.0178)	✓	0.4884 (0.0177)	✓	0.4931 (0.0171)	✓

Network 36:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-49.9564 (0.0213)		-49.9799 (0.0202)	✓	-49.9905 (0.0199)	✓
δ_1	-3	-3.0465 (0.0179)		-3.0269 (0.0170)	✓	-3.0067 (0.0169)	✓
δ_2	3	3.0311 (0.0175)	✓	3.0174 (0.0165)	✓	3.0077 (0.0162)	✓
$\Sigma_{\beta,11}$	0.5	0.5276 (0.0186)	✓	0.5199 (0.0179)	✓	0.5066 (0.0175)	✓
$\Sigma_{\beta,12}$	0.25	0.2559 (0.0147)	✓	0.2547 (0.0142)	✓	0.2532 (0.0140)	✓
$\Sigma_{\beta,22}$	0.5	0.5183 (0.0184)	✓	0.5153 (0.0173)	✓	0.5081 (0.0168)	✓

Network 37:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0578 (0.0191)		-50.0279 (0.0191)	✓	-50.0105 (0.0185)	✓
δ_1	-2	-2.0369 (0.0121)		-2.0196 (0.0120)	✓	-2.0144 (0.0117)	✓
δ_2	2	2.0374 (0.0117)		2.0157 (0.0117)	✓	2.0038 (0.0112)	✓
$\Sigma_{\beta,11}$	0.5	0.5087 (0.0120)	✓	0.5052 (0.0118)	✓	0.4970 (0.0112)	✓
$\Sigma_{\beta,12}$	0.25	0.2589 (0.0107)	✓	0.2564 (0.0103)	✓	0.2544 (0.0098)	✓
$\Sigma_{\beta,22}$	0.5	0.5137 (0.0119)	✓	0.5082 (0.0114)	✓	0.5026 (0.0114)	✓

Network 38:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-49.9635 (0.0214)	✓	-49.9729 (0.0194)	✓	-49.9870 (0.0168)	✓
δ_1	-2	-2.0302 (0.0119)		-2.0165 (0.0115)	✓	-2.0076 (0.0110)	✓
δ_2	3	3.0291 (0.0121)		3.0167 (0.0116)	✓	3.0114 (0.0116)	✓
$\Sigma_{\beta,11}$	0.5	0.4873 (0.0123)	✓	0.4914 (0.0120)	✓	0.4966 (0.0118)	✓
$\Sigma_{\beta,12}$	0.25	0.2593 (0.0093)	✓	0.2565 (0.0093)	✓	0.2497 (0.0091)	✓
$\Sigma_{\beta,22}$	0.5	0.4895 (0.0117)	✓	0.4925 (0.0112)	✓	0.5008 (0.0112)	✓

Network 39:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-49.9556 (0.0218)		-49.9761 (0.0210)	✓	-50.0127 (0.0185)	✓
δ_1	-3	-2.9719 (0.0124)		-2.9844 (0.0122)	✓	-2.9860 (0.0116)	✓
δ_2	2	2.0101 (0.0121)	✓	2.0076 (0.0115)	✓	1.9973 (0.0113)	✓
$\Sigma_{\beta,11}$	0.5	0.5062 (0.0120)	✓	0.5056 (0.0115)	✓	0.5048 (0.0112)	✓
$\Sigma_{\beta,12}$	0.25	0.2539 (0.0100)	✓	0.2537 (0.0096)	✓	0.2536 (0.0091)	✓
$\Sigma_{\beta,22}$	0.5	0.5089 (0.0130)	✓	0.5083 (0.0120)	✓	0.5081 (0.0116)	✓

Network 40:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0387 (0.0214)	✓	-50.0164 (0.0189)	✓	-50.0065 (0.0176)	✓
δ_1	-3	-3.0068 (0.0119)	✓	-3.0039 (0.0116)	✓	-2.9984 (0.0115)	✓
δ_2	3	3.0181 (0.0127)	✓	3.0133 (0.0121)	✓	3.0052 (0.0112)	✓
$\Sigma_{\beta,11}$	0.5	0.5077 (0.0117)	✓	0.5029 (0.0113)	✓	0.4995 (0.0114)	✓
$\Sigma_{\beta,12}$	0.25	0.2455 (0.0096)	✓	0.2467 (0.093)	✓	0.2474 (0.0093)	✓
$\Sigma_{\beta,22}$	0.5	0.5210 (0.0117)	✓	0.5171 (0.0112)	✓	0.5096 (0.0108)	✓

Network 41:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0202 (0.0240) ✓	-50.0077 (0.0212) ✓	-49.9949 (0.0188) ✓
δ_1	-2	-2.0553 (0.0185)	-2.0158 (0.0169) ✓	-2.0072 (0.0152) ✓
δ_2	2	2.0312 (0.0175) ✓	2.0196 (0.0171) ✓	2.0122 (0.0170) ✓
$\Sigma_{\beta,11}$	0.5	0.5178 (0.0202) ✓	0.5098 (0.0197) ✓	0.5015 (0.0159) ✓
$\Sigma_{\beta,12}$	0.25	0.2416 (0.0158) ✓	0.2422 (0.0158) ✓	0.2461 (0.0152) ✓
$\Sigma_{\beta,22}$	0.5	0.5093 (0.0192) ✓	0.5039 (0.0191) ✓	0.5010 (0.0181) ✓

Network 42:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9548 (0.0213)	-49.9692 (0.0209) ✓	-49.9956 (0.0204) ✓
δ_1	-2	-2.0331 (0.0185) ✓	-2.0277 (0.0163) ✓	-2.0040 (0.0151) ✓
δ_2	3	3.0519 (0.0200)	3.0310 (0.0192) ✓	3.0146 (0.0183) ✓
$\Sigma_{\beta,11}$	0.5	0.4829 (0.0182) ✓	0.4882 (0.0180) ✓	0.4928 (0.0174) ✓
$\Sigma_{\beta,12}$	0.25	0.2534 (0.0161) ✓	0.2513 (0.0157) ✓	0.2489 (0.0151) ✓
$\Sigma_{\beta,22}$	0.5	0.4929 (0.0172) ✓	0.4942 (0.0159) ✓	0.4962 (0.0152) ✓

Network 43:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0537 (0.0237)	-50.0314 (0.0222) ✓	-50.0140 (0.0199) ✓
δ_1	-3	-3.0149 (0.0182) ✓	-3.0098 (0.0182) ✓	-3.0049 (0.0173) ✓
δ_2	2	2.0470 (0.0173)	2.0031 (0.0171) ✓	2.0020 (0.0162) ✓
$\Sigma_{\beta,11}$	0.5	0.5136 (0.0182) ✓	0.5084 (0.0181) ✓	0.5017 (0.0176) ✓
$\Sigma_{\beta,12}$	0.25	0.2539 (0.0138) ✓	0.2537 (0.0134) ✓	0.2521 (0.0131) ✓
$\Sigma_{\beta,22}$	0.5	0.5024 (0.0176) ✓	0.5012 (0.0175) ✓	0.4997 (0.0168) ✓

Network 44:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9439 (0.0225)	-49.9622 (0.0218) ✓	-49.9877 (0.0193) ✓
δ_1	-3	-2.9634 (0.0184)	-2.9764 (0.0179) ✓	-2.9930 (0.0162) ✓
δ_2	3	3.0214 (0.0180) ✓	3.0112 (0.0168) ✓	3.0040 (0.0167) ✓
$\Sigma_{\beta,11}$	0.5	0.5176 (0.0175) ✓	0.5072 (0.0171) ✓	0.5048 (0.0170) ✓
$\Sigma_{\beta,12}$	0.25	0.2580 (0.0144) ✓	0.2514 (0.0137) ✓	0.2497 (0.0135) ✓
$\Sigma_{\beta,22}$	0.5	0.5075 (0.0182) ✓	0.5044 (0.0179) ✓	0.5031 (0.0178) ✓

Network 45:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0437 (0.0188)		-50.0226 (0.0181)	✓	-50.0077 (0.0173)	✓
δ_1	-2	-2.0242 (0.0118)		-2.0149 (0.0113)	✓	-2.0072 (0.0106)	✓
δ_2	2	1.9714 (0.0120)		1.9824 (0.0114)	✓	1.9940 (0.0114)	✓
$\Sigma_{\beta,11}$	0.5	0.5156 (0.0120)	✓	0.5072 (0.0117)	✓	0.5030 (0.0114)	✓
$\Sigma_{\beta,12}$	0.25	0.2610 (0.0102)	✓	0.2603 (0.0102)	✓	0.2578 (0.0098)	✓
$\Sigma_{\beta,22}$	0.5	0.5047 (0.0121)	✓	0.5029 (0.0114)	✓	0.5022 (0.0114)	✓

Network 46:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-49.9703 (0.0210)	✓	-49.9841 (0.0192)	✓	-49.9931 (0.0190)	✓
δ_1	-2	-1.9740 (0.0117)		-1.9833 (0.0110)	✓	-1.9863 (0.0110)	✓
δ_2	3	3.0214 (0.0115)	✓	3.0068 (0.0109)	✓	3.0037 (0.0107)	✓
$\Sigma_{\beta,11}$	0.5	0.4911 (0.0118)	✓	0.4932 (0.0115)	✓	0.5006 (0.0115)	✓
$\Sigma_{\beta,12}$	0.25	0.2685 (0.0101)	✓	0.2623 (0.0096)	✓	0.2535 (0.0095)	✓
$\Sigma_{\beta,22}$	0.5	0.4874 (0.0119)	✓	0.4886 (0.0117)	✓	0.4940 (0.0114)	✓

Network 47:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0393 (0.0185)	✓	-50.0309 (0.0176)	✓	-50.0271 (0.0156)	✓
δ_1	-3	-2.9951 (0.0112)	✓	-2.9954 (0.0110)	✓	-2.9975 (0.0108)	✓
δ_2	2	2.0156 (0.0112)	✓	2.0144 (0.0110)	✓	2.0144 (0.0109)	✓
$\Sigma_{\beta,11}$	0.5	0.4800 (0.0121)	✓	0.4801 (0.0114)	✓	0.4823 (0.0114)	✓
$\Sigma_{\beta,12}$	0.25	0.2293 (0.0098)		0.2383 (0.0094)	✓	0.2411 (0.0087)	✓
$\Sigma_{\beta,22}$	0.5	0.4889 (0.0120)	✓	0.4891 (0.0115)	✓	0.4893 (0.0112)	✓

Network 48:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-49.8488 (0.0207)	✓	-49.8961 (0.0193)	✓	-49.9916 (0.0178)	✓
δ_1	-3	-3.0358 (0.0118)		-3.0183 (0.0114)	✓	-3.0057 (0.0111)	✓
δ_2	3	3.0049 (0.0122)	✓	3.0037 (0.0118)	✓	3.0030 (0.0114)	✓
$\Sigma_{\beta,11}$	0.5	0.4894 (0.0114)	✓	0.4988 (0.0111)	✓	0.5005 (0.0110)	✓
$\Sigma_{\beta,12}$	0.25	0.2603 (0.0100)	✓	0.2573 (0.097)	✓	0.2549 (0.0094)	✓
$\Sigma_{\beta,22}$	0.5	0.4883 (0.0113)	✓	0.4908 (0.0110)	✓	0.4924 (0.0109)	✓

Network 49:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0377 (0.0233)	✓	-50.0258 (0.0222)	✓	-50.0088 (0.0207)	✓
δ_1	-2	-2.0436 (0.0184)		-2.0139 (0.0173)	✓	-2.0054 (0.0163)	✓
δ_2	2	2.0205 (0.0181)	✓	2.0148 (0.0170)	✓	2.0041 (0.0167)	✓
$\Sigma_{\beta,11}$	0.5	0.5207 (0.0197)	✓	0.5163 (0.0188)	✓	0.5034 (0.0172)	✓
$\Sigma_{\beta,12}$	0.25	0.2630 (0.0173)	✓	0.2588 (0.0164)	✓	0.2510 (0.0159)	✓
$\Sigma_{\beta,22}$	0.5	0.5129 (0.0188)	✓	0.5053 (0.0175)	✓	0.5037 (0.0171)	✓

Network 50:

	True value	5% of non-formed dyads sampled		10% of non-formed dyads sampled		15% of non-formed dyads sampled	
$log\lambda_0$	-50	-50.0422 (0.0241)	✓	-50.0319 (0.0212)	✓	-50.0186 (0.0203)	✓
δ_1	-2	-1.9640 (0.0193)	✓	-1.9811 (0.0186)	✓	-1.9866 (0.0171)	✓
δ_2	3	3.0311 (0.0186)	✓	3.0097 (0.0171)	✓	3.0056 (0.0167)	✓
$\Sigma_{\beta,11}$	0.5	0.5108 (0.0163)	✓	0.5055 (0.161)	✓	0.5042 (0.158)	✓
$\Sigma_{\beta,12}$	0.25	0.2396 (0.0157)	✓	0.2416 (0.0152)	✓	0.2424 (0.0140)	✓
$\Sigma_{\beta,22}$	0.5	0.5088 (0.0176)	✓	0.5040 (0.0164)	✓	0.5024 (0.0149)	✓

Network 51:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0119 ✓ (0.0214)	-50.0096 ✓ (0.0209)	-50.0057 ✓ (0.0200)
δ_1	-3	-3.0269 ✓ (0.0185)	-3.0124 ✓ (0.0172)	-3.0116 ✓ (0.0168)
δ_2	2	2.0304 ✓ (0.0182)	2.0085 ✓ (0.0169)	2.0038 ✓ (0.0162)
$\Sigma_{\beta,11}$	0.5	0.4933 ✓ (0.0190)	0.4947 ✓ (0.0184)	0.4988 ✓ (0.0181)
$\Sigma_{\beta,12}$	0.25	0.2335 ✓ (0.0136)	0.2349 ✓ (0.0134)	0.2379 ✓ (0.0129)
$\Sigma_{\beta,22}$	0.5	0.5213 ✓ (0.0183)	0.5135 ✓ (0.0172)	0.5096 ✓ (0.0171)

Network 52:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9509 (0.0219)	-49.9781 ✓ (0.0209)	-49.9825 ✓ (0.0197)
δ_1	-3	-3.0582 (0.0186)	-3.0272 ✓ (0.0184)	-3.0150 ✓ (0.0175)
δ_2	3	2.9538 (0.0182)	2.9771 ✓ (0.0175)	2.9935 ✓ (0.0171)
$\Sigma_{\beta,11}$	0.5	0.5229 ✓ (0.0182)	0.5127 ✓ (0.0172)	0.5083 ✓ (0.0169)
$\Sigma_{\beta,12}$	0.25	0.2311 ✓ (0.0153)	0.2426 ✓ (0.0132)	0.2452 ✓ (0.0120)
$\Sigma_{\beta,22}$	0.5	0.5079 ✓ (0.0183)	0.5071 ✓ (0.0181)	0.5022 ✓ (0.0178)

Network 53:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0293 (0.0194) ✓	-50.0112 (0.0185) ✓	-50.0030 (0.0175) ✓
δ_1	-2	-1.9786 (0.0117)	-1.9879 (0.0113) ✓	-1.9923 (0.0109) ✓
δ_2	2	2.0351 (0.0113)	2.0183 (0.0112) ✓	2.0086 (0.0107) ✓
$\Sigma_{\beta,11}$	0.5	0.4823 (0.0118) ✓	0.4905 (0.0118) ✓	0.4926 (0.0116) ✓
$\Sigma_{\beta,12}$	0.25	0.2607 (0.0097) ✓	0.2566 (0.0095) ✓	0.2521 (0.0093) ✓
$\Sigma_{\beta,22}$	0.5	0.4861 (0.0115) ✓	0.4946 (0.0112) ✓	0.4962 (0.0111) ✓

Network 54:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-49.9657 (0.0206) ✓	-49.9872 (0.0197) ✓	-49.9940 (0.0191) ✓
δ_1	-2	-2.0325 (0.0116)	-2.0184 (0.0115) ✓	-2.0065 (0.0111) ✓
δ_2	3	2.9833 (0.0111) ✓	2.9844 (0.0108) ✓	2.9953 (0.0106) ✓
$\Sigma_{\beta,11}$	0.5	0.5112 (0.0115) ✓	0.5076 (0.0109) ✓	0.5025 (0.0106) ✓
$\Sigma_{\beta,12}$	0.25	0.2318 (0.0098) ✓	0.2428 (0.0093) ✓	0.2454 (0.0092) ✓
$\Sigma_{\beta,22}$	0.5	0.5046 (0.0116) ✓	0.5028 (0.0114) ✓	0.5013 (0.0114) ✓

Network 55:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0417 (0.0201)	-50.0283 (0.0178) ✓	-50.0171 (0.0163) ✓
δ_1	-3	-2.9899 (0.0115) ✓	-3.0047 (0.0111) ✓	-2.9978 (0.0107) ✓
δ_2	2	1.9742 (0.0113)	1.9943 (0.0109) ✓	2.0016 (0.0108) ✓
$\Sigma_{\beta,11}$	0.5	0.5167 (0.0118) ✓	0.5069 (0.0117) ✓	0.5011 (0.0114) ✓
$\Sigma_{\beta,12}$	0.25	0.2404 (0.0101) ✓	0.2434 (0.0094) ✓	0.2441 (0.0091) ✓
$\Sigma_{\beta,22}$	0.5	0.4905 (0.0117) ✓	0.4926 (0.0113) ✓	0.4949 (0.0108) ✓

Network 56:

	True value	5% of non-formed dyads sampled	10% of non-formed dyads sampled	15% of non-formed dyads sampled
$log\lambda_0$	-50	-50.0397 (0.0206) ✓	-50.0254 (0.0196) ✓	-50.0047 (0.0181) ✓
δ_1	-3	-3.0083 (0.0114) ✓	-2.9973 (0.0112) ✓	-2.9997 (0.0109) ✓
δ_2	3	2.9628 (0.0119)	2.9813 (0.0114) ✓	2.9910 (0.0109) ✓
$\Sigma_{\beta,11}$	0.5	0.5065 (0.0116) ✓	0.4971 (0.0114) ✓	0.4992 (0.0112) ✓
$\Sigma_{\beta,12}$	0.25	0.2425 (0.0103) ✓	0.2482 (0.098) ✓	0.2514 (0.0093) ✓
$\Sigma_{\beta,22}$	0.5	0.5124 (0.0112) ✓	0.5049 (0.0107) ✓	0.5019 (0.0104) ✓

Overall Conclusions

We can make the following conclusions from the simulations:

- By sampling all the dyads that form ties and 10% (or 15%) of the dyads that do not form ties, we can accurately estimate parameter values using the WESBI method.
- The average posterior standard deviation in parameter estimates grows smaller as we sample more data.
- For both long- and short-tailed networks, parameter estimation using the WESBI method is equally good, and we observe the same patterns as we sample more data.

Estimation Time Advantage of the WESBI method

To assess the estimation time advantage of the WESBI method, we compare the average time taken by one iteration of the Bayesian inference procedure when the full dataset is used and when smaller sampled datasets are used. In the algorithm, the difference between the time taken by one iteration of the Bayesian inference procedure is very small regardless of whether the iteration belongs to burn-in phase or after the chains have converged. (The time taken is always within $\pm$9.3% of the average time taken by one iteration.) Thus we take the average across the first 1,000 iterations in the estimation procedure. In Table A5, we report the average time taken in seconds for completing the calculations of one iteration in the MCMC procedure of estimation from the full dataset and three different sampled datasets (sampling proportions are 5%, 10%, 15%, respectively) for each of the 56 networks we generated above. As we stated, the variance in time taken across iterations is small within each sampled dataset, so the reported numbers are suitable for comparison. The numbers in parentheses in the last three columns denote the percentage time taken as compared to using the full dataset.

Table A5: Iteration Times for WESBI for Different Sampling Proportions

	Time per iteration with full dataset	Time per iteration when 5% of non-tie dyads are sampled	Time per iteration when 10% of non-tie dyads are sampled	Time per iteration when 15% of non-tie dyads are sampled
Network 1	5.33	0.65 (12.3%)	0.88 (16.4%)	1.08 (20.2%)
Network 2	5.38	0.65 (12.1%)	0.88 (16.4%)	1.07 (20.0%)
Network 3	5.32	0.66 (12.4%)	0.88 (16.4%)	1.07 (20.1%)
Network 4	5.22	0.65 (12.4%)	0.86 (16.6%)	1.05 (20.0%)
Network 5	5.19	0.64 (12.3%)	0.84 (16.2%)	1.03 (19.8%)
Network 6	5.37	0.66 (12.3%)	0.88 (16.4%)	1.08 (20.1%)
Network 7	5.14	0.64 (12.4%)	0.84 (16.3%)	1.01 (19.7%)
Network 8	5.36	0.66 (12.3%)	0.88 (16.4%)	1.08 (20.1%)
Network 9	30.03	4.01 (13.4%)	5.28 (17.6%)	6.33 (21.1%)
Network 10	29.71	4.06 (13.7%)	5.10 (17.1%)	6.02 (20.3%)
Network 11	29.79	3.91 (13.1%)	5.16 (17.3%)	6.21 (20.8%)
Network 12	29.67	3.93 (13.3%)	5.03 (16.9%)	6.04 (20.4%)
Network 13	29.91	3.89 (13.0%)	5.25 (17.6%)	6.26 (20.9%)
Network 14	29.58	3.93 (13.3%)	4.94 (16.7%)	5.86 (19.8%)
Network 15	30.02	3.78 (12.6%)	5.30 (17.6%)	6.41 (21.3%)
Network 16	29.90	3.98 (13.3%)	5.19 (17.4%)	6.17 (20.6%)
Network 17	5.31	0.65 (12.3%)	0.88 (16.6%)	1.07 (20.2%)
Network 18	5.26	0.64 (12.2%)	0.87 (16.5%)	1.05 (19.9%)
Network 19	5.26	0.65 (12.3%)	0.86 (16.4%)	1.06 (20.2%)
Network 20	5.30	0.65 (12.3%)	0.87 (16.4%)	1.06 (19.9%)
Network 21	5.34	0.65 (12.2%)	0.87 (16.4%)	1.06 (19.9%)
Network 22	5.25	0.65 (12.4%)	0.87 (16.5%)	1.05 (20.1%)
Network 23	5.20	0.64 (12.3%)	0.85 (16.4%)	1.03 (19.8%)
Network 24	5.26	0.64 (12.2%)	0.85 (16.2%)	1.04 (19.9%)
Network 25	29.84	4.05 (13.6%)	5.14 (17.2%)	6.15 (20.6%)
Network 26	29.92	3.94 (13.2%)	5.18 (17.3%)	6.25 (20.9%)
Network 27	30.09	3.93 (13.1%)	5.33 (17.7%)	6.46 (21.5%)
Network 28	29.69	3.93 (13.3%)	5.02 (16.9%)	5.95 (20.1%)
Network 29	29.84	3.95 (13.2%)	5.15 (17.2%)	6.16 (20.6%)
Network 30	29.62	3.89 (13.1%)	5.05 (17.0%)	5.96 (20.1%)
Network 31	30.23	3.81 (12.6%)	5.45 (18.0%)	6.56 (21.7%)
Network 32	29.87	3.91 (13.1%)	5.14 (17.2%)	6.13 (20.5%)
Network 33	5.27	0.65 (12.3%)	0.86 (16.3%)	1.04 (19.8%)

Network 34	5.16	0.63 (12.3%)	0.84 (16.3%)	1.01 (19.6%)
Network 35	5.41	0.67 (12.3%)	0.90 (16.6%)	1.10 (20.3%)
Network 36	5.34	0.66 (12.3%)	0.89 (16.6%)	1.08 (20.2%)
Network 37	29.90	3.88 (13.0%)	5.21 (17.4%)	6.23 (20.8%)
Network 38	29.65	4.05 (13.7%)	5.03 (17.0%)	5.94 (20.0%)
Network 39	29.56	3.92 (13.3%)	4.99 (16.9%)	5.92 (20.0%)
Network 40	29.79	4.06 (13.6%)	5.12 (17.2%)	6.10 (20.5%)
Network 41	5.21	0.64 (12.3%)	0.85 (16.3%)	1.04 (20.0%)
Network 42	5.34	0.66 (12.3%)	0.88 (16.4%)	1.08 (20.2%)
Network 43	5.36	0.66 (12.2%)	0.89 (16.5%)	1.07 (20.0%)
Network 44	5.47	0.66 (12.2%)	0.90 (16.5%)	1.11 (20.3%)
Network 45	30.20	3.87 (12.8%)	5.35 (17.7%)	6.51 (21.6%)
Network 46	29.82	3.84 (12.9%)	5.15 (17.3%)	6.23 (20.9%)
Network 47	29.99	3.91 (13.0%)	5.26 (17.5%)	6.36 (21.2%)
Network 48	29.97	3.91 (13.0%)	5.25 (17.5%)	6.27 (20.9%)
Network 49	5.42	0.66 (12.3%)	0.89 (16.4%)	1.08 (20.0%)
Network 50	5.24	0.65 (12.4%)	0.87 (16.5%)	1.04 (19.8%)
Network 51	5.13	0.63 (12.3%)	0.84 (16.3%)	1.02 (19.9%)
Network 52	5.34	0.66 (12.3%)	0.88 (16.5%)	1.09 (20.5%)
Network 53	30.12	4.01 (13.3%)	5.37 (17.8%)	6.45 (21.4%)
Network 54	29.73	3.97 (13.3%)	5.09 (17.1%)	6.04 (20.3%)
Network 55	29.78	3.94 (13.2%)	5.11 (17.2%)	6.11 (20.5%)
Network 56	29.91	3.94 (13.2%)	5.25 (17.6%)	6.29 (21.0%)

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Online Technical Appendix B

Random Coefficients Model

Model Estimation

We extend our basic model to include individual-level heterogeneity. First, to capture unobserved heterogeneity in the baseline hazard rates across reviewers, we allow the parameter α_1 of the baseline hazard function to vary across senders using a log-normal distribution in the following way: $\lambda_{0,i}(t) = \alpha_0 \alpha_{1i} t^{\alpha_{1i}-1}$ and $\log(\alpha_{1,i}) \sim \mathrm{N}(\overline{\alpha_1}, \sigma_\alpha^2)$, where *i* is the index over senders. (Note that the heterogeneity in α_0 is absorbed by a_i, the sender-specific random effect.) Second, heterogeneity may exist because the same covariates may have different impacts on different reviewers' propensities to form trust relationships. To control for this, we allow for heterogeneity in the coefficients as follows: $\begin{bmatrix} \boldsymbol{\beta}_i^i \\ \boldsymbol{\beta}_i^j \\ \boldsymbol{\beta}_i^{ij} \end{bmatrix} = \boldsymbol{\beta}_i = \boldsymbol{\delta} + \boldsymbol{\varepsilon}_i,\ \boldsymbol{\varepsilon}_i \sim MVN(0, \boldsymbol{\Sigma}_\beta)$. The notation used here is similar to that used in the homogenous model in Section 3 of the paper. Let $\mathrm{C_{ij}}$ be the number of time periods for which dyad ij has been observed, and $\mathrm{T_{ij}}$ be the length of time from the starting point to the time period when i extends a tie to j. We define $\mathbb{I}_{\mathrm{ij}} = 1$ if $\mathrm{T_{ij}} \le \mathrm{C_{ij}}$ (i.e., if a tie formed within the observation time) and 0 otherwise, and $\mathrm{k_{ij}} = \mathrm{floor}\left(\min\left\{\mathrm{T_{ij}}, \mathrm{C_{ij}}\right\}\right)$. The log-conditional-likelihood function for this formulation is given by:

$$\log L = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \left\{ \mathbb{I}_{ij} \log\left[1 - \exp\left\{-\exp\left[\alpha_i(k_{ij}) + \boldsymbol{z}_{ij,k_{ij}} \boldsymbol{\beta_i} + \boldsymbol{a_i} + \boldsymbol{b_j} + \boldsymbol{d_{ij}}\right]\right\}\right] - \sum_{t=0}^{k_{ij}-1} \exp[\alpha_i(t) + \boldsymbol{z}_{ijt} \boldsymbol{\beta_i} + a_i + b_j + d_{ij}] \right\},$$

where $\alpha_i(t) = \ln\left(\int_t^{t+1} \lambda_{0,i}(u) du\right)$. The results for the model with heterogeneity are provided in the table below. We find that the impact of preferential attachment and recency are qualitatively the same as in the model with homogenous individuals.

Parameter Estimates for the "Movies" Category with Heterogeneous Coefficients

Variables	Posterior Mean	Posterior Std Deviation across Individuals
Receiver Characteristics		
Receiver's PrevAggReview	0.0774	0.5961***
Receiver's CurReview	0.4193***	0.2916***
Receiver's AggOpnLeadership	0.1789***	0.2587***
Receiver's CurOpnLeadership	0.3045***	0.4966***
Comprehensiveness	0.2351***	0.2437***
Objectivity	0.1157	0.2658***
Readability	0.1480	0.3049***
$(\text{Comprehensiveness})^2$	-0.2223***	0.5855***
$(\text{Objectivity})^2$	-0.0873***	0.1825***
$(\text{Readability})^2$	-0.3301***	0.4390***
Top Reviewer Label	0.1968***	0.3546***
Sender Characteristics		
Sender's AggReview	0.1477***	0.4048***
Sender's AggOutgoingLink	0.0888***	0.2001***
Dyad Characteristics		
Dissimilarity in Comprehnsiveness	-0.1445***	0.2035***
Dissimilarity in Objectivity	-0.1003**	0.1900***
Dissimilarity in Readability	-0.0739	0.6119***
Reciprocity	0.1941***	0.2138***
Number of Commonly Trusted Reviewers	0.1672***	0.4610***
Hazard Rate Parameters		
$\text{Log}(\alpha_0)$	-14.7143***	
$\overline{\alpha_1}$	-6.0919***	
σ_α^2	0.4977***	
σ_d^2	0.2078***	
σ_a^2	0.6080***	
σ_b^2	0.5282***	
$\boldsymbol{\sigma_{ab}}$	0.1379***	

***, ** and * denote that the 99% credible interval, the 95% credible interval, and the 90% credible interval, respectively, does not include zero.

Estimation Procedure

For the procedure described below, letters with superscript u represent the values of the corresponding updated parameters.

Step 1: $\boldsymbol{\beta}_i^u|\boldsymbol{\delta}, \Sigma_{\boldsymbol{\beta}}, a_i, b_i, \alpha_0, \alpha_{1,i}, d_{ij}, \text{data}$

$$f(\boldsymbol{\beta}_i^u|\boldsymbol{\delta}, \Sigma_{\boldsymbol{\beta}}, a_i, b_i, \alpha_0, \alpha_{1,i}, d_{ij}, \text{data})$$

$$\propto \mathrm{N}\left(\left(\boldsymbol{\beta}_i^u|\boldsymbol{\delta}, a_i, b_i, \alpha_0, \alpha_{1,i}, d_{ij}\right), \Sigma_{\boldsymbol{\beta}}\right) L(\boldsymbol{Y})$$

$$\propto \left|\Sigma_{\boldsymbol{\beta}}\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{\beta}_i^u - \boldsymbol{\delta})' \Sigma_{\boldsymbol{\beta}}^{-1}(\boldsymbol{\beta}_i^u - \boldsymbol{\delta})\right] L(\boldsymbol{Y})$$

where $L(\boldsymbol{Y})$ is the likelihood function. Since this distribution does not have a closed form, we use the Metropolis-Hastings algorithm to draw from the conditional distribution of $\boldsymbol{\beta_i}$. $\boldsymbol{\beta_i}$ is the draw of coefficients from the previous iteration, and we draw $\boldsymbol{\beta}_i^u$ by $\boldsymbol{\beta}_i^u = \boldsymbol{\beta_i} + \Delta\boldsymbol{\beta}$, where $\Delta\boldsymbol{\beta}$ is a draw from $\mathrm{N}(\mathbf{0}, \Delta^2\Lambda)$, and Δ and Λ are chosen adaptively to reduce the autocorrelation among the MCMC draws following Atchade (2006). The probability of accepting this $\boldsymbol{\beta}_i^u$, the updated value for $\boldsymbol{\beta_i}$ is:

$$\Pr(\text{acceptance}) = \min\{\frac{[\exp(-\frac{1}{2}\left((\boldsymbol{\beta}_i^u - \boldsymbol{\delta})' \Sigma_{\boldsymbol{\beta}}^{-1}(\boldsymbol{\beta}_i^u - \boldsymbol{\delta})\right)]L(\boldsymbol{Y}|\boldsymbol{\beta}_i^u)}{[\exp(-\frac{1}{2}\left((\boldsymbol{\beta}_i - \boldsymbol{\delta})' \Sigma_{\boldsymbol{\beta}}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\delta})\right)]L(\boldsymbol{Y}|\boldsymbol{\beta}_i)}, 1\}$$

Step 2: $\boldsymbol{\delta}^u|\Sigma_{\boldsymbol{\beta}}, \boldsymbol{\beta}_i^u$

$\boldsymbol{\delta}^u$ is generated from the distribution $\mathrm{MVN}(\boldsymbol{\mu_\delta}, \boldsymbol{V_\delta})$, where $\boldsymbol{\mu_\delta} = \boldsymbol{V_\delta}\left[\boldsymbol{\Sigma}_\beta^{-1} \sum_{i=1}^{N} \boldsymbol{\beta}_i^u + V_0^{-1}U_0\right]$, $\boldsymbol{V_\delta} = (\boldsymbol{N\Sigma}_\beta^{-1} + V_0^{-1})^{-\mathbf{1}}$. We define diffuse priors by setting $V_0 = 100I$ and $U_0 = 0$.

Step 3: $\Sigma_{\boldsymbol{\beta}}^u|\boldsymbol{\beta}_i^u, \boldsymbol{\delta}^u$

$$(\Sigma_{\boldsymbol{\beta}}^u|\boldsymbol{\beta}_i^u, \boldsymbol{\delta}^u) \sim \mathrm{IW}_{n(\beta)}(f_0 + N, \boldsymbol{G}_0^{-1} + \sum_{i=1}^{N}(\boldsymbol{\beta}_i^u - \boldsymbol{\delta}^u)(\boldsymbol{\beta}_i^u - \boldsymbol{\delta}^u)')$$

where we set $f_0 = n(\beta) + 5$ and $\boldsymbol{G_0} = I_{n(\beta)}$ to be diffuse hyperpriors. f_0 is the degrees of freedom, $\boldsymbol{G_0}$is the scale matrix of the inverse-Wishart distribution, and $n(\beta)$ is the number of δ parameters, the ones before observed covariates that we are interested in.

Step 4: $\alpha_{1,i}^{u}|\boldsymbol{\beta}_i^u, a_i, b_i, \alpha_0, \overline{\alpha_1}, \sigma_\alpha^2, d_{ij}, \text{data}$

We can define the distribution of $\alpha_{1,i}^{u}$ as:

$$f\left(\alpha_{1,i}^{u}\middle|\boldsymbol{\beta}_i^u, a_i, b_i, \alpha_0, \overline{\alpha_1}, \sigma_\alpha^2, d_{ij}, \text{data}\right)$$

$$\propto \text{N}((\alpha_{1,i}^{u}|\boldsymbol{\beta}_i^u, a_i, b_i, \alpha_0, \overline{\alpha_1}, d_{ij}), \sigma_\alpha^2)L(\boldsymbol{Y})$$

$$\propto \sigma_\alpha \exp\left[-\tfrac{1}{2}\left(\alpha_{1,i}^{u} - \overline{\alpha_1}\right)^2 \sigma_\alpha^{-2}\right] L(\boldsymbol{Y})$$

We use the Metropolis-Hastings algorithm to draw from the conditional distribution of a_i, a_i is the draw of coefficients from the previous iteration, and we draw $\alpha_{1,i}^{u}$ according to $\alpha_{1,i}^{u} = \alpha_{1,i} + \Delta\alpha$, where $\Delta\alpha$ is a draw from N(0,$\Delta^2\Lambda$), and Δ and Λ are chosen adaptively to reduce autocorrelation among MCMC draws following Atchade (2006),. The acceptance probability is:

$$\Pr(\text{acceptance}) = \min\{\frac{\left[\exp\left(-\frac{1}{2}\left(\alpha_{1,i}^{u} - \overline{\alpha_1}\right)^2 \sigma_\alpha^{-2}\right)\right] L\left(\boldsymbol{Y}\middle|\alpha_{1,i}^{u}\right)}{\left[\exp\left(-\frac{1}{2}\left(\alpha_{1,i} - \overline{\alpha_1}\right)^2 \sigma_\alpha^{-2}\right)\right] L\left(\boldsymbol{Y}\middle|\alpha_{1,i}\right)}, 1\}$$

Step 5: $\overline{\alpha_1}^u|\alpha_{1,i}^{u}, \sigma_\alpha^2, \text{data}$

$\overline{\alpha_1}^u$ is generated from a distribution $\text{N}(\mu_\alpha, \nu_\alpha)$, where $\mu_\alpha = \nu_\alpha\left[\sigma_\alpha^{-2}\sum_{i=1}^{N}\alpha_{1,i}^{u} + \nu_{\alpha_0}^{-1}U_0\right]$, $\nu_\alpha = (N\sigma_\alpha^{-2} + \nu_{\alpha_0}^{-1})^{-1}$. We define diffuse priors by setting $\nu_{\alpha 0} = 100$ and $U_0 = 0$.

Step 6: $(\sigma_\alpha^2)^u|\overline{\alpha_1}^u, \alpha_{1,i}^{u}$

$$\left((\sigma_\alpha^2)^u\middle|\overline{\alpha_1}^u, \alpha_{1,i}^{u}\right) \sim \text{InverseGamma}(f_0 + N, g_0^{-1} + \sum_{i=1}^{N}(\alpha_{1,i}^{u} - \overline{\alpha_1}^u)^2)$$

where we set $f_0 = 6$ and $g_0 = 1$ to be diffuse hyperprior. f_0 is the degrees of freedom, g_0 is the scale matrix of the inverse Gamma distribution.

Step 7: $\alpha_0^u|\boldsymbol{\beta}_i^u, a_i, b_i, \alpha_{1,i}^{u}, d_{ij}, \text{data}$

$$f(\alpha_0^u|\boldsymbol{\beta}_i^u, a_i, b_i, \alpha_{1,i}^{u}, d_{ij}, \text{data}) \propto \sigma_{\alpha_0}^{-1}\exp[-\frac{1}{2}(\alpha_0^u - \overline{\alpha_0})^2\sigma_{\alpha_0}^{-2}]L(\boldsymbol{Y})$$

where $\overline{\alpha_0}$ and $\sigma_{\alpha 0}^2$ are diffuse priors. Because there is no closed form for this, we use the Metropolis-Hastings algorithm to draw from this conditional distribution of α_0^u . The probability of accepting α_0^u is:

$$\Pr(\text{acceptance}) = \min\{\frac{\left[\exp\left(-\frac{1}{2}(\alpha_0^u - \overline{\alpha_0})^2\sigma_{\alpha_0}^{-2}\right)\right]L(\boldsymbol{Y}|\alpha_0^u)}{\left[\exp\left(-\frac{1}{2}(\alpha_0 \ - \overline{\alpha_0})^2\sigma_{\alpha_0}^{-2}\right)\right]L(\boldsymbol{Y}|\alpha_0\ )}, 1\}$$

We define diffuse priors by setting $\overline{\alpha_0} = 0$ and $\sigma_{\alpha_0}^2$=30.

Step 8: Generate a_i^u, b_i^u:

$$f(a_i^u, b_i^u|\boldsymbol{\beta}_i^u, \Sigma_{ab}, \alpha_0^u, \alpha_{1,i}^u, d_{ij}, \text{data})$$

$$\propto \mathrm{N}\left((a_i^u, b_i^u|\boldsymbol{\beta}_i^u, \alpha_0^u, \alpha_{1,i}^u, d_{ij}), \Sigma_{ab}\right)L(\boldsymbol{Y})$$

$$\propto |\Sigma_{ab}|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}(a_i^u, b_i^u)\Sigma_{ab}^{-1}(a_i^u, b_i^u)'\right]L(\boldsymbol{Y})$$

Because this distribution does not have a closed form, we use the Metropolis-Hastings algorithm to draw from the conditional distribution of a_i, b_i: a_i, b_i is the draw of the random effect from the previous iteration, and we draw a_i^u, b_i^u by $\begin{bmatrix} a_i^u \\ b_i^u \end{bmatrix} = \begin{bmatrix} a_i \\ b_i \end{bmatrix} + \Delta\begin{bmatrix} a \\ b \end{bmatrix}$, where $\Delta\begin{bmatrix} a \\ b \end{bmatrix}$ is a draw from N(0,$\Delta^2\Lambda$), and Δ and Λ are chosen adaptively to reduce autocorrelation among MCMC draws following Atchade (2006). The probability of accepting this $\begin{bmatrix} a_i^u \\ b_i^u \end{bmatrix}$, the updated value for $\begin{bmatrix} a_i \\ b_i \end{bmatrix}$ is:

$$\Pr(\text{acceptance}) = \min\{\frac{\left[\exp\left(-\frac{1}{2}(a_i^u, b_i^u)\Sigma_{ab}^{-1}(a_i^u, b_i^u)'\right)\right]L(\boldsymbol{Y}|a_i^u, b_i^u)}{\left[\exp\left(-\frac{1}{2}(a_i\ , b_i\ )\Sigma_{ab}^{-1}(a_i\ , b_i\ )'\right)\right]L(\boldsymbol{Y}|a_i\ , b_i\ )}, 1\}$$

Step 9: $\Sigma_{\mathrm{ab}}^{\mathrm{u}}|a_i^u, b_i^u$

$$(\Sigma_{\mathrm{ab}}^{\mathrm{u}}|a_i^u, b_i^u) \sim IW_2(7 + N, G_0^{-1} + \sum_{i=1}^{N}(a_i^u, b_i^u)(a_i^u, b_i^u)')$$

Step 10: $d_{ij}^u, d_{ji}^u|\alpha_0^u, \boldsymbol{\beta}_i^u, a_i, b_i, \alpha_{1,i}^u, \sigma_d^2, \text{data}$

$$f(d_{ij}^u, d_{ji}^u|\alpha_0^u, \boldsymbol{\beta}_i^u, a_i, b_i, \alpha_{1,i}^u, \sigma_d^2, \text{data})$$

$$\propto \mathrm{N}\left((d_{ij}^u, d_{ji}^u|\alpha_0^u, \boldsymbol{\beta}_i^u, a_i, b_i, \alpha_{1,i}^u), \sigma_d^2\right)L(\boldsymbol{Y})$$

$$\propto \sigma_d^{-1}\exp\left[-\frac{1}{2}\left(d_{ij}^u + d_{ji}^u\right)^2\sigma_d^{-2}\right]L(\boldsymbol{Y})$$

We use the Metropolis-Hastings algorithm to draw from this conditional distribution of d_{ij}^u and d_{ji}^u: d_{ij} and d_{ji} are the draw of the unobservable similarity effects from the previous iteration, and we draw d_{ij}^u, d_{ji}^u by $\begin{bmatrix} d_{ij}^u \\ d_{ji}^u \end{bmatrix} = \begin{bmatrix} d_{ij} \\ d_{ji} \end{bmatrix} + \Delta\boldsymbol{d}$, where $\Delta\boldsymbol{d}$ is a draw from N(0,$\Delta^2\Lambda$), and Δ and Λ are chosen adaptively to reduce autocorrelation among MCMC draws following Atchade (2006). The probability of accepting $\begin{bmatrix} d_{ij}^u \\ d_{ji}^u \end{bmatrix}$ is:

$$\Pr(\text{acceptance}) = \min\{\frac{\left[\exp\left(-\frac{1}{2}\left(d_{ij}^u + d_{ji}^u\right)\sigma_d^{-2}\right)\right] L(\boldsymbol{Y}|d_{ij}^u, d_{ji}^u)}{\left[\exp\left(-\frac{1}{2}\left(d_{ij} + d_{ji}\right)\sigma_d^{-2}\right)\right] L(\boldsymbol{Y}|d_{ij}, d_{ji})}, 1\}$$

Step 11: Generating σ_d^u

$$(\sigma_d^u |\, d_{ij}^u, d_{ji}^u) \sim \mathrm{IW}_1(1 + N(N-1), 1 + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (d_{ij}^u + d_{ji}^u)^2)$$

Step 12: If convergence is not yet reached, go to Step 1.